:- Probability :-

The Theory of Probability is the Science of Providing a numerical measures of Uncertainity that prevails various events that we encounter in our everyday life.

Probability Theory had its origin in The games of chance. Gamblers used it earlier, to find The most probable chance cases, in cases of different games of chance.

RANDOM EXPRIMENT :- By a random experiment we mean an experiment whose all possible outcomes are known and which can be repeated under identical conditions but exact Prediction of the out Come in any Particular trial is impossible.

For example if we toss a Coin we get a random experiment. Because here either a head or a tail can occure, but exact Prediction is any tossing is impossible.

TRIAL AND EVENTS:-

Performance of random experiment is called a trial and the results obtained from it are called events.

For example (i) Tossing of a Coin is a example of a trial and getting of head or tail is called event.

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Bayes Theorem :-

Let $E_1, E_2, \ldots E_n$ are mutually disjoint events with $P(E_i) \neq 0 \ (i = 1, 2, \ldots n)$. Then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, then we have

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \quad ; i = 1, 2, \ldots n \longrightarrow (1)$$

Proof :-

Since $A \subset \bigcup_{i=1}^{n} E_i$, we have

$A = A \cap (\bigcup_{i=1}^{n} E_i) = \bigcup_{i=1}^{n} (A \cap E_i)$ / By distributive

Since $(A \cap E_i) \subset E_i, \ i = 1, 2, \ldots n$ are mutually disjoint events, we have by addition theorem of Probability (or Axiom 3 of Probability)

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] = \sum_{i=1}^{n} P(A \cap E_i)$$

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i) \longrightarrow (*)$$

Now by Compound Theorem of Probability.

Also we have

$$P(A \cap E_i) = P(A)\cdot P(E_i/A)$$

$$\Rightarrow P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\cdot P(A/E_i)}{P(A)}$$

$$\Rightarrow P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \quad \text{from } eq^n (*)$$

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Applications:

(1.) The probabilities $P(E_1), P(E_2), \cdots P(E_n)$ are termed as the "a priori probabilities" because they exist before we gain any information from the experiment itself.

(2.) The probabilities $P(A/E_i); i=1,2,\cdots n$ are called likelyhood because they indicate how likely the event 'A' under consideration is to occure, given each and every priori probability.

(3.) Probabilities $P(E_i/A); i=1,2,\cdots n$, are called Posterior Probabilities because they are determined after the results of the experiment are known.

(4.) ~~From $P(A) = P[P(A \cap E_i)] = P(A)\,P(E_i/A)$~~

(5.) From $P(A) = P[\bigcup_{i=1}^{n}(A \cap E_i)] = \sum_{i=1}^{n} P(A \cap E_i) = \sum P(E_i)P(A/E_i)$

We get the following important results.

If the events $E_1, E_2, \cdots E_n$ constitute a partition of the sample space S and $P(E_i) \neq 0$, $i=1,2,\cdots n$ then for any event A in S we have

$$P(A) = \sum_{i=1}^{n} P(A \cap E) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i)$$

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Random Variable :-

The various out comes of a random experiment is denoted by the help of a variable which is called a random variable (R.V.). For example: In case of Throwing a die, we may use a variable X for representing the out come of Throw. Thus X will take the values 1, 2, 3, 4, 5, 6.

But in some cases the out comes may be qualitative e.g. in tossing a coin which may be head or tail; the colour of balls drawn from an Urn may be red, yellow, White etc. But for mathematical convinience the qualitative out comes may be expressed in quantitative form.

For example in tossing a coin we may denote the out come "Head" by 1 and "Tail" by 0. In this way each out come of a random experiment, whether it is qualitative or quantitative, can be expressed by a real number.

The real number, which is associated with the out comes of a random experiment is called a random Variable.

Def^n :-

Def^n of R.V :- A real valued function X, defined on a sample space S, of a random experiment is called a random variable which assigns to each sample point, one and only one real number

$$X(s) = x \text{ (say)} \quad \text{where } s \in S.$$

Here S denotes the sample space corresponding to a random experiment.

Different types of Random Variable:

There two types of random variable,

a) Continuous random variable
b) Discreate random Variable

a) Continuous random Variable :- If a random variable is such that it assumes any value with in a given interval, Then it is called as a Continuous random Variable. In other words if a random variable can take infinite numbers of values within a given interval $a \le x \le b$ (say) then it is called a Continuous rand Variable

Example: Height of the persons collected from a Crowd

b) Discreate random variable :-

If a random variable X assumes only a finite numbers or Countably infinite numbers of values, Then it is called a discreate random variable.

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The random variable x is said to take a finite values only if the possible values of x are $x_1, x_2, \ldots x_n$ and is said to be Countably infinite if x takes the values $x_1, x_2, \ldots$

Example: Number of throws of a fair coin before the first head occurs.

Probability Distribution:-

The distribution obtained by listing the possible values of a random variable with their respective probabilities is called a probability distribution. A probability distribution can be presented either with the help of a function or in tabular form where values of the random variable and corresponding probability are shown.

The probability distribution for a discrete r.v. is called a discrete probability distribution or Probability mass function (p.m.f.) and that of a continuous probability distribution for a continuous r.v. is called a continuous probability distribution or Probability density function (p.d.f)

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Geometric Distribution :-

~~Let us consider a~~

A random variable X is said to have a geometric distribution, if it assumes ~~only~~ only non-negative values and its P.m.f. is given by

$$P(X=x) = \begin{cases} q^x p, & x = 0, 1, 2, \cdots \quad 0 < p < 1 \\ 0, & \text{otherwise} \end{cases} \longrightarrow (1)$$

Note: 1) Since the various probabilities for $x = 0, 1, 2, \cdots$ are the various terms of geometric progression, hence the name geometric distribution.

2.) Clearly, assignment of ~~probabilities~~ probabilities in eq^n (1) is permissible, since

$$\sum_{x=0}^{\infty} P(X=x) = \sum_{x=0}^{\infty} q^x p = p(1 + q + q^2 + q^3 + \cdots)$$

$$= p(1-q)^{-1}$$

$$= \frac{p}{(1-q)} = 1$$

~~Negative Binomial distribution~~ :-

~~A random variable~~

Mean and Variance of Geometric distribution :-

Here we have

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} x q^x p \qquad \left[\because 1 - 2q + 3q^2 + 4q^3 \cdots = (1-q)^{-2} \right]$$

$$= p(q + 2q^2 + 3q^3 + \cdots)$$

$$= pq(1 + 2q + 3q^2 + \cdots) \qquad [\because p + q = 1 \;\; \therefore p = 1 - q]$$

$$= pq(1-q)^{-2} = \frac{pq}{(1-q)^2} = \frac{pq}{p^2}$$

$= \frac{q}{p}$

$\Rightarrow$ Mean of Geometric distribution is $\frac{q}{p}$

Now

$$E(x^2) = \sum_{x=0}^{\infty} x^2 P(X=x) = \sum_{x=0}^{\infty} x^2 q^x p$$

$$= qp + 4q^2p + 9q^3p + 16q^4p + \cdots$$

$$= qp(1 + 4q + 9q^2 + 16q^3 + \cdots)$$

$$= \frac{q(q+1)}{(1-q)^2} = \frac{q(q+1)}{p^2}$$

$p+q=1$

$p=1-q$

$$\therefore E(x^2) = \frac{q(q+1)}{p^2}$$

$$\therefore \text{variance} = E(x^2) - \{E(x)\}^2$$

$$= \frac{q(q+1)}{p} - \frac{q^2}{p^2} = \frac{q^2+q-q^2}{p^2}$$

$$= q/p^2$$

$\therefore$ Variance of Geometric dist^n is $= \frac{q}{p^2}$

$q(1-q^2)\cdot(1-q)^{-3}$

Negative Binomial Distributions :-

Negative Binomial Distributions :-

A random Variable 'x' is said to follow a negative binomial distribution, if its p.m.f. is given by

$$p(x) = P(X=x) = \binom{x+n-1}{n-1} p^n q^x \; ; \; x = 0, 1, 2, \ldots \longrightarrow (1)$$

$= 0$, otherwise. Where 'x' is an Integere

$0 < p < 1$, n and p are two parameters

Also, $\binom{x+n-1}{n-1} = \binom{x+n-1}{x}$ $\quad \left[\because \binom{n}{r} = \binom{n}{n-r}\right]$

$$= \frac{(x+n-1)(x+n-2)\cdots\cdots(n+1)n}{x!}$$

$$= \frac{(-1)^x(-n)(-n-1)\cdots\cdots(-n-x+2)(-n-x+1)}{x!}$$

$$= (-1)^x \binom{-n}{x}$$

$$\therefore p(x) = P(X=x) = \binom{-n}{x} p^n (-q)^x \; ; \; x = 0, 1, 2, \ldots \longrightarrow (2)$$

$= 0$, otherwise.

Which is the $(x+1)^{th}$ terem in the expansion of $p^n(1-q)^{-n}$, a binomial ~~distribution~~ expansion with a negative Index. Hence the distribution is known as negative binomial distribution.

N.B: One of the most Important Characteristics of the negative Binomial dist^n is That

Mean of Negativ binomial dist^n < Variance of Negative binomial dist^n

ie In Negative Binomial dist^n

Mean & Variance.

Mean $= rp$

Variance $= rpq$

Mean and Variance of Negative distribution

The M.G.F. of Negative Binomial dist^n is defined as:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} q^{-r} \left(-\frac{pe^t}{q}\right)^x$$

$$= (q - pe^t)^{-r} \longrightarrow (3)$$

Here $p = \frac{1}{q}$, $q = p/q$

So that $q - p = 1$

$\therefore p + q = 1$

Now, $\mu_1' = \left(\frac{d}{dt} M_x(t)\right)_{t=0}$

$$= \left[-r(-pe^t)(q - pe^t)^{-r-1}\right]_{t=0}$$

$$= rp$$

$\therefore$ Mean of Negativ Binomial dist^n is rp

Now

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} \longrightarrow (4)$$

$$= \left(rpe^t(q - pe^t)^{-r-1} - (-r-1)rpe^t(q - pe^t)^{-r-2}(0 - pe^t)\right.$$

$$= rp + r(r+1)p^2$$

$$\therefore \mu_2 = \mu_2' - (\mu_1')^2$$

$$= n(n+1)p^2 + np - (np)^2$$

$$= n^2p^2 + np^2 + np - n^2p^2$$

$$= np(p+1) = npq.$$

$\therefore$ If $\mu_2 \Rightarrow$ variance $= npq$. ⟶ (5)

[$\because p - q = 1$ because $p+q=1$; $p = \frac{1}{q}$ So that $q = p/q$. $q - p = 1$

$p+q = \frac{1}{q} + \frac{p}{q}$

$= \frac{p+1}{q}$

$= \frac{q}{q} = 1$]

Another Method:-

The mean and Variance of Negative Binomial distribution Through p.g.f Method we have

$$P(X = x, n) = \binom{x+n-1}{n-1} p^n q^x \; ; \; n = 0, 1, 2, \ldots$$

Now $P(S) = PGF$

$$= E(S^x) = \sum_{n=0}^{\infty} \binom{x+n-1}{n-1} p^n (qs)^x.$$

$$= \sum_{n=0}^{\infty} \binom{x+n-1}{n-1} p^n (qs)^x$$

$$= p^n \sum_{n=0}^{\infty} \binom{x+n-1}{n-1} (qs)^x = p^n (1-sq)^{-n}.$$

$\therefore$ Mean $\Rightarrow \mu_1' = \left[\frac{dP(s)}{ds}\right]_{s=1} = p^n x - n(1-sq)^{-n-1} x (-sq)\Big]_{s=1}$.

$$= p^n n p^{-n-1} q = nq \longrightarrow (*)$$

Now $\left[\frac{d^2P(s)}{ds^2}\right]_{s=1} = nqp^n(-n-1)(1-sq)^{-n-2}(-q)\Big]_{s=1}$

$$= nq^2p^n(n-1)(1-sq)^{-n-2}\Big]_{s=1}$$

P.T.O

$$= r(r-1)q^2 p^r p^{r-2}$$

$$= \frac{r(r-1)q^2}{p^2} \neq \longrightarrow (**)$$

$\therefore$ Variance $(X) \Rightarrow \mu_2$

$$= \frac{d^2 p(s)}{ds^2}\Big]_{s=1} + \frac{dp(s)}{dp(s)}\Big]_{s=1} - \left[\left(\frac{dp(s)}{ds}\right)\right.$$

$p+q=1$
$1-q=p$
$1-q=1$
$\frac{p}{p}$

$\frac{p-q}{p}=0$

$$= \frac{r(r-1)q^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{r(r-1)q^2 + rpq - r^2q^2}{p^2}$$

$$= \frac{r^2q^2 - rq^2 + rpq - r^2q^2}{p^2}$$

$$= \frac{rpq - rq^2}{p^2} = \frac{rq(p-q)}{p^2}$$

$$(*) = \frac{rq(p-q)}{p^2} = \frac{rq}{p}\left(1 - \frac{q}{p}\right)$$

$$= \frac{rq}{p}\left[\frac{(r-1)q}{p} + 1 - \frac{rq}{p}\right]$$

$$= \frac{rq}{p}\left[\frac{rq - q}{p} + p - rq\right] \quad \Big| p+q=1$$

$$= \frac{rq}{p}\left[1 - \frac{q}{p}\right] = \frac{rq}{p^2}(p-q)$$

$$= \frac{rq}{p^2}(p-q)$$

$$\text{Var}(X) = \frac{nq}{p^2}n(n-1)q^2 + \frac{nq}{p} - \frac{n^2q^2}{p^2}$$

$$= \frac{nq}{p}\left[\frac{(n-1)q}{p} + 1 - \frac{nq}{p}\right]$$

$$= \frac{nq}{p}\left[\frac{nq - q + p - nq}{p}\right]$$

$$= \frac{nq}{p^2}\left[(p-q)\right]$$

Since $0 < p < 1$ Therefore $(p-q)$ is a Constant Where $p-q \neq 0$ So $p-q$ will be a Constant.

(*) $\text{Var}(X) = \mu_2 = \left.\frac{d^2P(s)}{ds^2}\right|_{s=1} + \left.\frac{dP(s)}{ds}\right|_{s=1} - \left(\left.\frac{dP(s)}{ds}\right|_{s=1}\right)^2$

$$= \frac{n(n-1)q^2}{p^2} + \frac{nq}{p} - \frac{n^2q^2}{p^2}$$

$$= \frac{nq}{p^2}\left[(n-1)q + p - nq\right] = \frac{nq}{p^2}\left[nq - q + p - nq\right]$$

$$= \frac{nq}{p^2}\left[nq - q + p - nq\right]$$

$$= \frac{nq}{p^2}\left[p - q\right] \quad \because p \neq 0 \quad \therefore p-q \text{ is a Constant}$$

$q \to 0$

$$= \frac{nq}{p^2} \times \text{Constant} \text{ is } > \left(\frac{nq}{p}\right).$$

$\therefore$ Mean & Variance of a Negative Binomial dist^n.

Uniform Distribution

Let X be a continuous random variable assuming all values in the interval (a, b) where a and b are finite values. If the P.d.f of X is given by

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$
$$= 0, \text{ otherwise}$$

Then the distribution is said to be distributed uniformly over the interval (a, b) and the distribution is called as uniform distribution. The uniform distribution is also termed as rectangular distribution.

Cumulative Distribution function:

The cumulative distribution function or CDF of a random variable X is given by

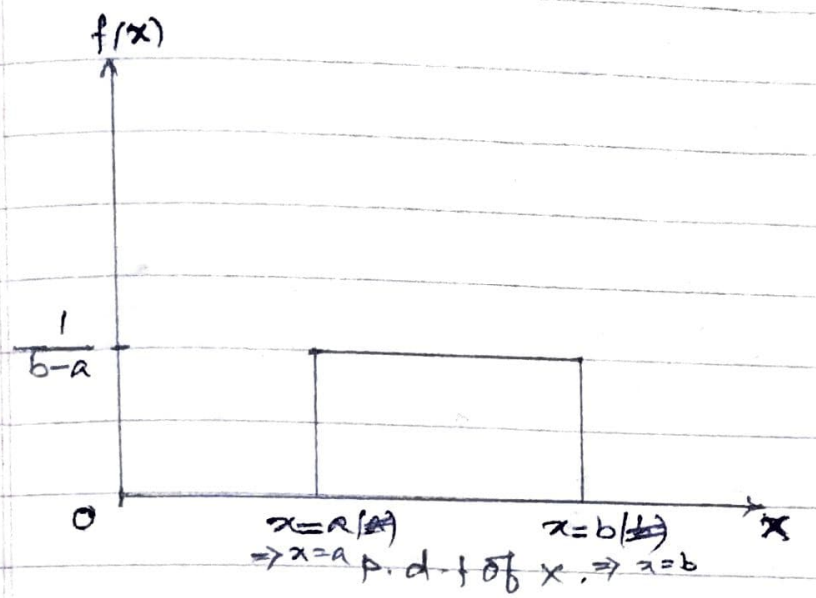
$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= \int_{a}^{x} \frac{dx}{b-a} = \frac{1}{b-a}\left[x\right]_{a}^{x}$$

$$= \frac{x-a}{b-a}$$ Thus we have, $F(x) =$

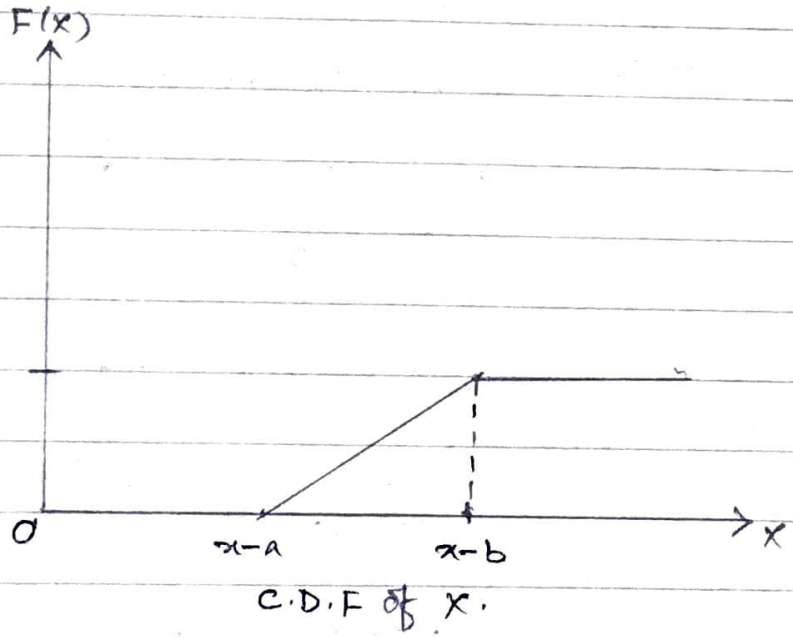
$$= \frac{x-a}{b-a} \text{ if } a \le x \le b \text{ and if } x <$$

Thus The graph of the p.d.f of 'x' is given by,



$(x-b)-(x-a)$
$= x-b-x+a$
$= a-b = -(b-a)$.

$-[x-b]+[x-a]$
$= x-b+x+a$
$= a-b$



C.D.F of X.

Remarks:- (1) / a and b (a < b) are the two

Mean and Variance of the distribut

We have $E(x) = \int_a^b x f(x)\,dx$.

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \left[\frac{x^2}{2(b-a)}\right]_a^b \Rightarrow \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{b^2-a^2}{2(b-a)} \qquad = \frac{b+a}{2}$$

$\therefore E(x) = \frac{b+a}{2}$

$$E(x^2) = \int_a^b x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b = \frac{1}{b-a} \times \frac{b^3-a^3}{3}$$

$$= \frac{1}{(b-a)} \times \frac{(b-a)(b^2+ab+a^2)}{3}$$

$$= \frac{1}{3}(b^2+ab+a^2)$$

Now $Var(x) = E(x^2) - [E(x)]^2$

$$= \frac{1}{3}(b^2+ab+a^2) - \frac{(b+a)^2}{4}$$

$$= \frac{1}{12}\left[4b^2+4ab+4a^2-3b^2-6ab-3a^2\right)$$

$$= \frac{1}{12}(a^2 - 2ab + b^2) = \frac{1}{12}(a-b)^2$$

∴ Mean of Uniform distribution is $= \frac{b+a}{2}$.
and Variance of Uniform distribution is $\frac{1}{12}(a-b)^2$

— x —

Remarks:

① a and b $(a < b)$ are the two parameters of the Uniform distribution on (a, b)

② The distribution is also known as rectangular dist^n Since the curve $y = f(x)$ describes a rectangl over the x-axis and between the ordinates at $x = a$ and $x = b$.

③ The distribution function $F(x)$ is given by:

$$F(x) = \begin{cases} 0, & \text{if } -\infty < x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b < x < \infty \end{cases}$$

Since $F(x)$ is not continuous at $x = a$ and $x = b$, it is not differentiable at these points. Thus $\frac{d}{dx} F(x) = f(x) = \frac{1}{b-a} \neq 0$, exists every where except at the points $x = a$ and $x = b$.

$$P(X=x) = \frac{{}^{N_p}C_x \times {}^{N_q}C_{n-x}}{{}^{N}C_n}, \quad x = 0, 1, 2, \ldots, n$$

Hyper Geometric Distribution:-

When a population is finite and the sampling is done without replacement, so that the events are stochastically independent, although random, we obtain hypergeometric distribution. Let an urn contain N balls, M of which are white and N-M are red. Suppose that we draw a sample of 'n' balls at random (without replacement) from the urn, Then the probability of getting 'k' white balls out of 'n' ($k \leq n$) is

$$\binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n}$$

$$\Rightarrow P(X=x) = h(k, N, M, n) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

Where $k = 0, 1, 2, \ldots, \min(n, M)$

$= 0$, otherwise. $\longrightarrow (1)$

Definition:-

A discrete random variable x is said to follow the hypergeometric distribution if it assumes only non-negative values and its p.m.f (Probability mass function) is given by

P. T. O

$$P(X=k) = h(k, N, M, n)$$

$$= \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \quad ; k = 0, 1, 2, \ldots \rightarrow (1)$$

$$= 0, \text{ otherwise}$$

Note:- (1) N, M and n are known as parameters of hypergeometric distribution.

(2) As it can be shown that

$$\sum_{k=0}^{n} \binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n} = 1.$$

This assignment of probabilities is permissible

Mean and Variance of hypergeometric distribution:-

We know That

$$\text{Mean} = E(x) = \sum_{k=0}^{n} k\,p(x=k)$$

$$= \sum k \left\{ \binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n} \right\}$$

$$= \frac{M}{\binom{N}{n}} \sum_{k=1}^{n} \left\{ \binom{M-1}{k-1}\binom{N-M}{n-K} \right\}$$

$$= \frac{M}{\binom{N}{n}} \sum_{k=1}^{n} \binom{A}{x}\binom{N-A-1}{m-x}$$

Where $x = k-1$, $m = n-1$, $M-1 = A$.

$m - x$
$= n-1-K+1$
$= n-K$

$\binom{N-A-1}{m-n}$

$\Rightarrow \binom{N-1}{m}$

$$= \frac{M}{\binom{N}{n}}\binom{N-1}{m} = \frac{M}{\binom{N}{n}}\binom{N-1}{n-1}$$

$$= \frac{M}{\frac{N!}{n!(N-n)!}} \cdot \frac{(N-1)!}{(n-1)!(N-n)!} = \frac{M\,n!(N-n)!(N-1)!}{N\cdot(N-1)!\,(n-1)!(N-n)!}$$

$$= \frac{M}{N}\left[\frac{n(n-1)!(N-n)!(N-1)!}{(N-1)!(n-1)!(N-n)!}\right] = \frac{Mn}{N}$$

$\therefore E(x) = \frac{nM}{N} \longrightarrow (*)$ P.T.O

Now. $E\{x(x-1)\} = \sum_{k=0} k(k-1)\left\{\binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n}\right\}$

$= \frac{M(M-1)}{\binom{N}{n}} \sum_{k=2}^{n}\left\{\binom{M-2}{k-2}\binom{N-M}{n-k}\right\}$

$= \frac{M(M-1)}{\binom{N}{n}} \binom{N-2}{n-2}$

$= \frac{M(M-1)}{N!} \frac{(N-2)!}{(n-2)!(N-n)!}$

$\frac{}{n!(N-n)!}$

$= \frac{M(M-1)(N-n)!\, n(n-1)(n-2)!\,(N-2)!}{N(N-1)(N-2)!\qquad (n-2)!\,(N-n)!}$

$= \frac{M(M-1)\cdot n(n-1)}{N(N-1)} = \frac{(M^2-M)(n^2-n)}{(N^2-N)}$

$= \frac{M^2n^2 - M^2n - Mn^2 + Mn}{(N^2-N)}$

$\therefore Var(x) = E\{x(x-1)\} + E(x) - \{E(x)\}^2$

i.e. $Var(x) = E(x^2) - \{E(x)\}^2$

$\therefore E\{x(x-1)\} + E(x) = \frac{M^2n^2 - M^2n - Mn}{(N^2-N)} + \frac{nM}{N}$

$\therefore Var(x) = E(x^2) - \{E(x)\}^2 = \left(\frac{nM}{N}\right)^2$

$$= \frac{NM^2n^2 - NM^2n - NMn + N^2nM - NMn}{N(N^2-N)}$$

$$\therefore \text{Var}(X) = \frac{NM^2n^2 - NM^2n - NMn + N^2nM - NMn}{N^2(N^2-1)} - \frac{n^2M^2}{N^2}$$

$$= \frac{N^2M^2n^2 - N^2M^2n - N^2Mn - N^2nM - N^2n^2M^2 + n^2M^2}{N^2(N^2-1)}$$

$$= \frac{NM(NMn - Nn - N^2n - Nn) - n^2M^2}{N^2(N-1)}$$

$$= \frac{NM[Nn(M-1) - Nn(N-1)}{}$$

$$= \frac{}{N^2(N-1)}$$

$$E\{x(x-1)\} + E(X) = \frac{M^2n^2 - N^2n - Mn^2 + Mn}{N^2(N^2-N)} + \frac{Mn}{N}$$

$$= \frac{M^2n^2 - M^2n - Mn^2 + Mn}{N^2(N-1)} + \frac{Mn}{N}$$

$$= \frac{M^2n(n-1) - Mn(n-1)}{N^2(N-1)} + \frac{Mn}{N}$$

$$= \frac{M(M-1)\cdot n(n-1)}{N^2(N-1)} + \frac{Mn}{N}$$

$$\therefore \text{Var}(X) = E\{x(x-1)\} + E(X) - [E(X)]^2$$

$$= \frac{M(M-1)n(n-1)}{N^2(N-1)} + \frac{Mn}{N} - \left(\frac{Mn}{N}\right)^2$$

$$= \frac{NM(M-1)n(n-1) + N(N-1)Mn - (N-1)M^2n^2}{N^2(N-1)N^2}$$

Exponential Distribution :-

A Continuous random variable x assuming non-negative values is said to have exponential distribution with parameter $\theta > 0$, if it p.d.f is given by:

$$f(x) = \theta \cdot e^{-\theta x}, \quad x \geq 0 \longrightarrow (1)$$
$$= 0 ; \text{ Otherwise}$$

The cumulative distribution function of exponential distribution is $F(x)$ is given by

$$F(x) = \int_0^x f(u)\,du = \theta \int_0^x \exp(-\theta u)\,du$$

Here $F(x) = 1 - \exp(-\theta x) ; \quad x \geq 0 \longrightarrow 1(a)$
$$= 0, \text{ Otherwise}$$

Mean and Variance of exponential dist^n :-

The Moment generating function of exponential distribution is

$$M_x(t) = E(e^{tx}) = \theta \int_0^\infty e^{tx} e^{-\theta x}\,dx$$

$$= \theta \int_0^\infty e^{-(\theta - t)x}\,dx = \frac{\theta}{\theta - t} ; \theta > t$$

$$= \left[1 - \frac{t}{\theta}\right]^{-1} = \left(1 - \frac{t}{\theta}\right)^{-1}$$

$$= \sum_{r=0}^{\infty} \left(\frac{t}{\theta}\right)^r$$

$\therefore \mu_n' = E(x^n) = \text{Coefficient of } \frac{t^n}{n!} \text{ in } M_x(t)$

$= \frac{n!}{\theta^n} \; ; n = 1, 2, \ldots$

$\therefore \mu_1' = \frac{1}{\theta^1} = \frac{1}{\theta}$

Again Variance $\Rightarrow \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$

$= \frac{2}{\theta^2} - \frac{1}{\theta^2} = 1/\theta^2$

$\therefore$ Mean $= 1/\theta$ and Variance $= 1/\theta^2$

— x —

Gamma Distribution :- The Continuous Random Variable x which is distributed according to the Probability Law :

$$f(x) = \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} \; ; \text{where } \lambda > 0, \; 0 < x < \infty \qquad \rightarrow (2)$$

$= 0$, otherwise

is known as gamma variate with parameter λ and it is referred to as a $\gamma(\lambda)$ variate and its distribution is known as gamma distribution.

Mean and Variance of Gamma distribution:

We know that the M.G.F. of Gamma dist about origin is given by

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx.$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\,dx.$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \cdot \frac{\Gamma(\lambda)}{(1-t)^\lambda}; \quad |t|<1.$$

$$\Rightarrow M_x(t) = (1-t)^{-\lambda}; \quad |t|<1. \longrightarrow (3)$$

∴ The Cumulant generating function $K_x(t)$ is given by

$$K_x(t) = \log M_x(t) = \log(1-t)^{-\lambda} \quad <1.$$

$$\Rightarrow K_x(t) \Rightarrow -\lambda \log(1-t)$$

$$= -\lambda\left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots\right)$$

$$\therefore K_x(t) = \lambda\left(t + t^2/2 + t^3/3 + t^4/4 + \cdots\right)$$

∴ Mean = K_1 = Coefficient of t in $K_x(t)$ is λ.

Variance = K_2 = Coefficient t^2 in $K_x(t)$ is λ.

∴ Mean and variance of Gamma dist^n is $= \lambda$.

Beta Distribution of First kind :-

The Continuous random variable which is ~~distributed~~ distributed according to the probability law

$$f(x) = \begin{cases} \frac{1}{B(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1}; & (\mu,\nu)>0,\ 0<x<1 \\ =0, & \text{otherwise} \end{cases}$$

where $B(\mu,\nu)$ is the Beta function, is known as a Beta variate of the first kind with parameters μ and ν and is referred to as $\beta_1(\mu,\nu)$ variate and its distribution is called Beta distribution of the first kind.

Note:- ① The Cumulative distribution function, often called the Incomplete Beta Function is

$$F(x) = \begin{cases} 0, & x<0 \\ \int_0^1 \frac{1}{B(\mu,\nu)} u^{\mu-1}(1-u)^{\nu-1}\,du; & 0<x<1,\ (\mu,\nu)>0 \\ 1, & x>1 \end{cases} \longrightarrow (1)$$

② In particular, if we take $\mu=1$, and $\nu=1$ in eq^n. (1) we get

$$f(x) = \frac{1}{\beta(1,1)} = 1, \quad 0<x<1 \longrightarrow (1)(a)$$

which is the p.d.f. of Uniform distribution on $(0,1)$

③ if $x \sim \beta_1(\mu,\nu)$, then it can be easily proved that

$$1-x \sim \beta_1(\nu,\mu).$$

Beta Distribution of Second Kind :-

The Continuous random Variable X which is distributed according to the Probability law :

$$f(x) = \begin{cases} = \frac{1}{B(\mu,\nu)} \cdot \frac{x^{\mu-1}}{(1+x)^{\mu+\nu}}, & (\mu,\nu) > 0 \\ & 0 < x < \infty \\ = 0, & \text{otherwise} \end{cases} \longrightarrow (2)$$

is known as a Beta variate of Second kind parameters μ and ν and is denoted by as Variate and its distribution is called Beta distribution of Second kind.

Remark :- Beta distribution of second is transformed to Beta distribution of first kind by the transformation

$$1+x = \frac{1}{y} \Rightarrow y = \frac{1}{1+x} \longrightarrow (*)$$

Thus if $x \sim \beta_2(\mu,\nu)$, then y defined in (*) is $\beta_1(\mu,\nu)$.

— x —

Binomial Distribution:-

Def^n:- A discrete random variable X is said to follow Binomial distribution, if its p.m.f is given by

$$p(x) = {}^nC_x p^x q^{n-x} \quad ; \quad x = 0, 1, 2, \ldots n$$

$q = 1-p$ i.e $p+q=1$.

Here n and p are called the parameters of this dist^n.

Note:- (i) if X follows a Binomial Distribution with parameters n and p then it is usually denoted by

$$X \sim B(n, p) \text{ or } b(x, n, p)$$

(ii) Since the probability of getting $0, 1, 2, \ldots n$ successes i.e

${}^nC_0 p^0 q^n, {}^nC_1 p q^{n-1}, {}^nC_2 p^2 q^{n-2}, \ldots {}^nC_n p^n q^0$ are the successive terms of $(q+p)^n$, a binomial expansion. Therefore the distribution is called the Binomial distribution.

Mean of Binomial dist^n:

We have $E(x) = \sum_{x=0}^{n} x p(x) = \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$

$$= \sum x \frac{n}{x} \cdot {}^{n-1}C_{x-1} p^{x-1} \cdot q^{n-x}$$

[if Since ${}^nC_x = \frac{n}{x} {}^{n-1}C_{x-1}$; if $x=0$ then ${}^{n-1}C_{x-1} = {}^{n-1}C_{-1}$ is meaningless.]

Therefore the lower limit of the summation sign is taken as $x=1$]

P.T.O

$$\therefore E(x) = \sum_{x=0}^{n} x \, {}^{n}C_x p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n}{x} \cdot {}^{n-1}C_{x-1} \, p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{n-x}$$

$$= np(p+q)^{n-1} = np.$$

$\therefore$ Mean $= np$

Similarly the Variance of Binomial distribution

$$\text{Var}(x) = E(x^2) - \{E(x)\}^2 \longrightarrow (A)$$

Where

$$E(x^2) = \{E\{x(x-1) + x\}$$

$$= E\{x(x-1)\} + E(x) \longrightarrow (*)$$

Here $E\{x(x-1)\} = \sum_{x=0}^{n} x(x-1) \, p(x)$

$$= \sum_{x=0}^{n} x(x-1) \, {}^{n}C_x \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \left[x(x-1) \frac{n(n-1)}{x(x-1)} \, {}^{n-2}C_{x-2} \, p^n p^{x-2} q^{n-x} \right]$$

$$= n(n-1)p \sum_{x=2}^{0} {}^{n-2}C_{x-2} \, p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{n-2} q^{n-x}$$

$= n(n-1)p^2 (p+q)^{n-2} = n(n-1)p^2$

$\therefore E(x^2) = E\{x(x-1)\} + E(x) = n(n-1)p^2 + np = n^2p^2 - np^2 + np$

$\therefore \text{Var}(x) = E(x^2) - \{E(x)\}^2$

$= n(n-1)p^2 + np - n^2p^2$

$= (n^2 - n)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1-p) = npq.$

$\therefore$ The variance of Binomial distribution is $= npq.$

Note: (i) $\because q < 1 \Rightarrow npq < np$

i.e. Var < Mean.

For binomial distribution the mean is greater than the variance. which is an important characteristics of this binomial distribution.

(ii) The S.D of Binomial distribution is

$S.D. = \sqrt{\text{Var}(x)} = \sqrt{\text{npq}}$

—x—

Poisson Distribution:-

efination:- A discreat R.V. X is said to follow distribution is its p.m.f is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, 2, \ldots$$

here λ is called the parameter of this distn
(After the name of the French Mathe
S. D. Poisson This distn is called the poisson distribution.

Poisson distribution is a limiting form of Binomial distribution.

The poisson distribution can be derived as a limiting form of Binomial di under the following condition:-

(i) 'n' the number of independent Bernoulli trial is very large ie $n \to \infty$

(ii) P, the probability of success in each trial very small ie. $p \to 0$

(iii) $np = \lambda$ (say) a finite number.

Here we know that the p.m.f of Binomial d

is $P(X=x) = {}^nC_x p^x q^{n-x}; \quad x = 0, 1, 2, \ldots n$

$= \frac{n!}{x!(n-x)!} p^x q^{n-x}$

Here if we supposed that -

$np = \lambda$, $\therefore p = \lambda/n$ and $q = 1-p = 1-\lambda/n$

$$\therefore P(X=x) = \frac{n(n-1)(n-2)(n-3)\cdots(n-x-1)}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\lambda/n\right)^{n-x}$$

Now $\lim_{n\to\infty} P(X=x) = \frac{1}{x!}\lim_{n\to\infty}\left[\frac{n(n-1)(n-2)\cdots(n-x-1)}{x!(n-x)!}\right]\frac{\lambda^x}{n^x}\left(1-\lambda/n\right)^{n-x}$

[$\because$ from $(n-0)$ to $(n-x-1)$ we have x factores]
$\therefore$ power of n is x.

$$\therefore \lim_{n\to\infty} P(X=x) = \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\frac{n(n-1)(n-2)\cdots(n-x-1)!}{n^x}\right]\left(1-\lambda/n\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\frac{n\cdot n(1-1/n)\cdot n(1-2/n)\cdots n(1-\frac{x-1}{n})}{n^x}\right]\left(1-\lambda/n\right)^{n-x} \longrightarrow (1)$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\frac{n^x(1-1/n)(1-2/n)\cdots(1-\frac{x-1}{n})}{n^x}\right]\left(1-\lambda/n\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\lambda/n\right)^{n-x}\cdot$$

[$\because \lim_{n\to\infty}(1-1/n) = 1$ and so on.]

Here $\lim_{n\to\infty}\left(1-\lambda/n\right)^{n-x}$

$= \lim_{n\to\infty}\left(1-\lambda/n\right)^{n}\lim_{n\to\infty}\left(1-\lambda/n\right)^{-x}$

$= \bar{e}^{\lambda}\cdot 1$

[$\lim_{n\to\infty}(1-\lambda/n)^{\lambda} = \bar{e}^{\lambda}$

$\lim_{n\to\infty}(1-\lambda/n)^{-x} = 1$.]

Now putting the value of $\lim_{n\to\infty}\left(1-\lambda/n\right)^{n-x}$ in eq^n no (1) we get :

$$\lim_{n\to\infty} p(x=x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left[\frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}\right.$$

$$= \lim_{n\to\infty} \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\therefore \lim_{n\to\infty} p(x=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

Which is p.m.f. of the poisson distribution

Mean and Variance of Poisson dist^n:-

$$\text{Mean} \Rightarrow E(x) = \sum_{x=0}^{\infty} x\, p(x=x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{\lambda \cdot \lambda^{x-1} \cdot e^{-\lambda}}{x\cdot(x-1)!}$$

$$= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \lambda$$

$\therefore$ Mean of poisson dist^n is $= \lambda$.

Again Variance $= E(x^2) - [E(x)]^2$

Here

$$E\{x(x-1)+x\} = E(x^2)$$

$$\therefore E\{x(x-1)\} = \sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$= \sum x(x-1)\frac{\lambda^2 \cdot \lambda^{x-2} \cdot e^{-\lambda}}{x(x-1)(x-2)!}$$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} \cdot e^{-\lambda}}{(x-2)!} = \lambda^2$$

$$\therefore \text{Var}(x) = E\{x(x-1)\} + E(x) - \{E(x)\}^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

∴ The variance of Poisson distribution is $= \lambda$.

∴ In poisson distribution Mean = variance $= \lambda$.