

§ 1·2. Finite differences.

Suppose we have a function $y=f(x)$, where x can take the values $a, a+h, a+2h, \dots, a+nh$. Let the corresponding values of y i.e. $f(x)$ be $f(a), f(a+h), f(a+2h), \dots, f(a+nh)$. The expression $f(a+h) - f(a)$ is called the first difference of $f(a)$ and we denote this difference by $\Delta f(a)$.

Thus $\Delta f(a) = f(a+h) - f(a)$.

Similarly $f(a+2h) - f(a+h), \dots, f(a+nh) - f(a+(n-1)h)$ are all called the first differences and are generally denoted as

$$\Delta f(x) = f(x+h) - f(x), \quad x = a, a+h, \dots, a+(n-1)h. \quad \dots(1)$$

Here Δ is an operator and is called, a forward difference operator or simply the difference operator, h is called the interval of differencing.

If we operate Δ on (1), we get second differences of the function-values.

Thus $\Delta \{\Delta f(x)\} = \Delta \{f(x+h) - f(x)\}$

or $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

or $\Delta^2 f(x) = \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}, \text{ from (1)}$
 $= f(x+2h) - 2f(x+h) + f(x).$

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Here Δ^2 represents that the operation of differences has been done twice.

We may continue to define third differences as

$$\begin{aligned}\Delta^3 f(x) &= \Delta \Delta^2 f(x) = \Delta [f(x+2h) - 2f(x+h) + f(x)] \\&= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x) \\&= \{f(x+3h) - f(x+2h)\} - 2 \{f(x+2h) - f(x+h)\} \\&\quad + \{f(x+h) - f(x)\} \\&= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)\end{aligned}$$

and so on.

In general the n th forward difference is given by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x).$$

From above we observe that each higher difference can be expressed in terms of the preceding differences.

§ 1.3. Difference table.

The various differences obtained when shown in the form of a table is called a difference table. It is as given below :

Argument x	Entry $f(x)$	First differences $\Delta f(x)$	Second differences $\Delta^2 f(x)$
a	$f(a)$	$f(a+h) - f(a)$ $= \Delta f(a)$	
$a+h$	$f(a+h)$	$f(a+2h) - f(a+h)$ $= \Delta f(a+h)$	$\Delta f(a+h) - \Delta f(a)$ $= \Delta^2 f(a)$
$a+2h$	$f(a+2h)$	$f(a+3h) - f(a+2h)$ $= \Delta f(a+2h)$	$\Delta f(a+2h) - \Delta f(a+h)$ $= \Delta^2 f(a+h)$
$a+3h$	$f(a+3h)$	$f(a+4h) - f(a+3h)$ $= \Delta f(a+3h)$	$\Delta f(a+3h) - \Delta f(a+2h)$ $= \Delta^2 f(a+2h)$
$a+4h$	$f(a+4h)$:	:
:	:	:	:

The first entry $f(a)$ is called the *leading term* and the topmost differences in each column are called *leading differences*.

Ex. 1. Given $f(0)=3, f(1)=12, f(2)=81, f(3)=200, f(4)=100$ and $f(5)=8$. Form a difference table and find $\Delta^6 f(0)$.

[Gorakhpur 1982]

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Sol. The difference table is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	3					
1	12	9				
2	81	69	60	-10		
3	200	119	50	-269	-259	
4	100	-100	-219	227	496	755
5	8	-92	8			

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Argument x	Entry $f(x)$	First differences $\Delta f(x)$	Second differences $\Delta^2 f(x)$
a	$f(a)$		
$a+h$	$f(a+h)$	$f(a+h)-f(a) = \Delta f(a)$	$\Delta f(a+h)-\Delta f(a) = \Delta^2 f(a)$
$a+2h$	$f(a+2h)$	$f(a+2h)-f(a+h) = \Delta f(a+h)$	$\Delta f(a+2h)-\Delta f(a+h) = \Delta^2 f(a+h)$
$a+3h$	$f(a+3h)$	$f(a+3h)-f(a+2h) = \Delta f(a+2h)$	$\Delta f(a+3h)-\Delta f(a+2h) = \Delta^2 f(a+2h)$
$a+4h$	$f(a+4h)$	$f(a+4h)-f(a+3h) = \Delta f(a+3h)$	
\vdots	\vdots	\vdots	\vdots

The first entry $f(a)$ is called the *leading term* and the topmost differences in each column are called *leading differences*.

Ex. 1. Given $f(0)=3$, $f(1)=12$, $f(2)=81$, $f(3)=200$, $f(4)=100$ and $f(5)=8$. Form a difference table and find $\Delta^5 f(0)$.

[Gorakhpur 1982]

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Sol. The difference table is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	3					
1	12	9	60	-10		
2	81	69	50	-269	-259	
3	200	119	-219	227	496	755
4	100	-100	8			
5	8	-92				

Hence $\Delta^5 f(0)=755$.

Ex. 1 (a). If

x :	1	2	3	4	5
y :	2	5	10	20	30

find by forward difference table $\Delta^4 f(1)$. (Meerut 1991 P)

Sol. The forward difference table is as follows :

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	2	3			
2	5	5	2	3	
3	10	10	5	-5	-8
4	20	10	0		
5	30				

From table we observe that $\Delta^4 f(1) = -8$.

Ex. 1 (b). If $f(0) = -3, f(1) = 6, f(2) = 8$ and $f(3) = 12$, prepare forward difference table to find $\Delta^3 f(0)$. (Meerut 1991)

Sol. Proceed as above. $\Delta^3 f(0) = 9$.

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§ 1·4. Backward difference operator ∇

The operator ∇ defined by $\nabla f(x) = f(x) - f(x-h)$, is called the backward difference operator.

The second backward difference of $f(x)$ denoted by $\nabla^2 f(x)$ is the backward difference of the first backward difference $\nabla f(x)$.

$$\begin{aligned} \text{i.e. } \nabla^2 f(x) &= \nabla \nabla f(x) = \nabla \{ f(x) - f(x-h) \} \\ &= \nabla f(x) - \nabla f(x-h) \\ &= \{ f(x) - f(x-h) \} - \{ f(x-h) - f(x-2h) \} \\ &= f(x) - 2f(x-h) + f(x-2h) \end{aligned}$$

and so on.

In general the n th backward difference of $f(x)$ is given by

$$\nabla^n f(x) = \nabla^{n-1} [\nabla f(x)] = \nabla^{n-1} [f(x) - f(x-h)]$$

§ 1·5. Central difference operator δ .

The operator δ defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right),$$

is called the central difference operator.

§ 1·6. Identity operator I.

The operator I defined by $If(x) = f(x)$, is called the identity operator. The identity operator I is also often denoted by the symbol 1.

§ 1·7. The shifting operator E.

The operator E defined by $Ef(x) = f(x+h)$, where h is the interval of differencing, is called the shifting operator or the translation. By applying the operator E twice, we have

$$E^2 f(x) = EE f(x) = Ef(x+h) = f(x+2h),$$

$$E^3 f(x) = EE^2 f(x) = Ef(x+2h) = f(x+3h),$$

and so on.

$$\text{In general } E^n f(x) = f(x+nh).$$

Also we define E^{-1} by

$$E^{-1} f(x) = f(x-h) \text{ so that } E^{-n} f(x) = f(x-nh).$$

§ 1·8. Properties of the operators E and Δ .

(i) The operators Δ and E are linear operators :

$$\begin{aligned} \Delta [f_1(x) + f_2(x)] &= \{f_1(x+h) + f_2(x+h)\} - \{f_1(x) + f_2(x)\} \\ &= \{f_1(x+h) - f_1(x)\} + \{f_2(x+h) - f_2(x)\} \\ &= \Delta f_1(x) + \Delta f_2(x) \end{aligned}$$

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$$\begin{aligned} \text{and } \Delta [af(x)] &= af(x+h) - af(x) \\ &= a[f(x+h) - f(x)] = a\Delta f(x), \end{aligned}$$

therefore Δ is a linear operator.

Similarly we can show that E is a linear operator.

(ii) The operators Δ and E are commutative in operation with respect to constants :

If c is a constant, then

$$\Delta [cf(x)] = cf(x+h) - cf(x) = c\Delta f(x)$$

$$\text{and } E [cf(x)] = cf(x+h) = cEf(x).$$

(ii) The operators Δ and E are commutative in operation with respect to constants :

If c is a constant, then

$$\Delta \{c f(x)\} = cf(x+h) - cf(x) = c \Delta f(x)$$

and $E \{c f(x)\} = cf(x+h) = c E f(x).$

Note. The operators Δ and E are not commutative with respect to the functions of x i.e. (variables)

i.e. if $u_x = f(x) \cdot g(x)$, then $\Delta u_x \neq f(x) \cdot \Delta g(x)$

and $E u_x \neq f(x) \cdot E g(x).$

(iii) The operators Δ and E are distributive : We have

$$\Delta [f_1(x) + f_2(x) + \dots] = \Delta f_1(x) + \Delta f_2(x) + \dots$$

and $E [f_1(x) + f_2(x) + \dots] = E f_1(x) + E f_2(x) + \dots$

(iv) The operators Δ and E are commutative :

$$\begin{aligned} (\Delta E) f(x) &= \Delta [E f(x)] = \Delta f(x+h) \\ &= f(x+2h) - f(x+h) = E f(x+h) - E f(x) \\ &= E [f(x+h) - f(x)] = (E \Delta) f(x). \end{aligned}$$

Thus $\Delta E = E \Delta.$

(v) The operators Δ and E are associative :

$$(\Delta E) \Delta f(x) = \Delta (E \Delta) f(x).$$

(vi) The operators Δ and E obey the law of indices :

$$\begin{aligned} \Delta^m \Delta^n f(x) &= (\Delta \Delta \dots m \text{ times}) (\Delta \Delta \dots n \text{ times}) f(x) \\ &= [\Delta \Delta \dots (m+n) \text{ times}] f(x) \\ &= \Delta^{m+n} f(x). \end{aligned}$$

Similarly $E^m E^n f(x) = E^{m+n} f(x).$

(vii) Difference of the product of two functions.

$$\Delta [f(x) g(x)] = [E f(x)] \cdot \Delta g(x) + g(x) \cdot \Delta f(x).$$

We have $\Delta [f(x) \cdot g(x)] = f(x+h) \cdot g(x+h) - f(x) \cdot g(x),$

by definition of Δ

$$= f(x+h) g(x+h) - f(x+h) g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)$$

[Note]

$$= f(x+h) [g(x+h) - g(x)] + g(x) [f(x+h) - f(x)]$$

$$= f(x+h) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$$

$$= E f(x) \cdot \Delta g(x) + g(x) \cdot \Delta f(x).$$

(viii) Difference of the quotient of two functions :

$$\Delta \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \Delta f(x) - f(x) \cdot \Delta g(x)}{g(x) \cdot E g(x)}.$$

$$\begin{aligned}
 &= \frac{f(x+h) \cdot g(x) - f(x) \cdot g(x+h)}{g(x) \cdot g(x+h)} \\
 &= \frac{[f(x+h) - f(x)] g(x) - [g(x+h) - g(x)] f(x)}{g(x) \cdot g(x+h)} \quad [\text{Note}] \\
 &= \frac{g(x) \cdot \Delta f(x) - f(x) \cdot \Delta g(x)}{g(x) \cdot E g(x)}.
 \end{aligned}$$

(ix) $\Delta = E - I$ or $E = I + \Delta$.

We have

$$\begin{aligned}
 \Delta f(x) &= f(x+h) - f(x) \\
 &= Ef(x) - If(x) = (E - I)f(x).
 \end{aligned}$$

Therefore $\Delta = E - I$.

$$\begin{aligned}
 \text{Also } Ef(x) &= f(x+h) = \{f(x+h) - f(x)\} + f(x) \\
 &= \Delta f(x) + If(x) = If(x) + \Delta f(x) = (I + \Delta)f(x)
 \end{aligned}$$

so that $E = I + \Delta$.

(x) $\Delta^2 = E^2 - 2E + I$.

$$\begin{aligned}
 \text{We have } \Delta^2 f(x) &= \Delta \Delta f(x) = \Delta [f(x+h) - f(x)] \\
 &= \Delta f(x+h) - \Delta f(x) \\
 &= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\} \\
 &= f(x+2h) - 2f(x+h) + f(x) \\
 &= E^2 f(x) - 2Ef(x) + If(x) \\
 &= [E^2 - 2E + I]f(x).
 \end{aligned}$$

$\therefore \Delta^2 = E^2 - 2E + I$.

Similarly, $\Delta^3 = E^3 - 3E^2 + 3E - I$.

(Meerut 1991 P)

§ 1.9. Fundamental theorem of the difference calculus.

If $f(x)$ be a polynomial of n^{th} degree in x , then the n^{th} difference of $f(x)$ is constant and $\Delta^{n+1} f(x) = 0$.

Proof. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, where n is a positive integer and a_0, a_1, \dots, a_n are constants. Then by definition of Δ , we have

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x) \\&= [a_0 + a_1(x+h) + a_2(x+h)^2 + \dots + a_n(x+h)^n] \\&\quad - [a_0 + a_1x + a_2x^2 + \dots + a_nx^n] \\&= a_1h + a_2[(x+h)^2 - x^2] + \dots + a_n[(x+h)^n - x^n] \\&= a_1h + a_2[{}^2C_1 xh + h^2] + \dots \\&\quad \dots + a_n[{}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + {}^nC_{n-1}h^n] \\&= b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-2} + nh a_n x^{n-1}, \quad \dots(1)\end{aligned}$$

where $b_0, b_1, b_2, \dots, b_{n-1}$ are constant coefficients.

Thus the first difference of a polynomial of degree n is again a polynomial of degree $n-1$.

Repeating the operation we will observe that $\Delta \Delta f(x)$ i.e. $\Delta^2 f(x)$ is a polynomial of degree $(n-2)$ with last term as $n(n-1)h^2 a_n x^{n-2}$.

Continuing this process n times we will get a polynomial of degree $n-n$ i.e. of degree zero with the last and the only term as $n(n-1)(n-2)\dots 1 \cdot h^n a_n x^{n-n} = n! h^n a_n$.

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Thus $\Delta^n f(x) = n! h^n a_n$ which is a constant and hence $\Delta^{n+1} f(x) = 0$ and so all higher differences of order $> n$ are also zero.

Note. The converse of the above theorem is also true i.e. if the n^{th} difference of a function is constant then it may be expressed as a polynomial of degree n .

Solved Examples

Ex. 1. If Δ and ∇ be the first descending difference operator and first ascending difference operator respectively of function $f(x)$ show that $(\Delta - \nabla) \equiv \Delta \nabla$. [Agra 1988]

Sol. We have $\Delta \nabla f(x) = \Delta [f(x) - f(x-h)]$
 $= \Delta f(x) - \Delta f(x-h) = \{f(x+h) - f(x)\} - \{f(x) - f(x-h)\}$
 $= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x).$

Hence $\Delta \nabla \equiv \Delta - \nabla$.

Ex. 2. Evaluate the following :

(i) $\Delta(x^2 + e^x + 2)$.

[Meerut 1991 P]

Sol. By definition $\Delta f(x) = f(x+h) - f(x)$.

$$\therefore \Delta(x^2) = (x+h)^2 - x^2 = 2hx + h^2;$$

$$\Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1);$$

and $\Delta(2) = 2\Delta(1) = 2.(1-1) = 2.0 = 0$.

Hence $\Delta(x^2 + e^x + 2)$

$$= \Delta(x^2) + \Delta(e^x) + \Delta(2)$$

$$= 2hx + h^2 + e^x(e^h - 1) + 0.$$

(ii) $\Delta^2(ab^{cx})$

$$= \Delta[\Delta ab^{cx}] = \Delta[ab^{c(x+h)} - ab^{cx}]$$

$$= \Delta[ab^{cx}(b^{ch}-1)]$$

$$= \Delta[a(b^{ch}-1)b^{cx}]$$

$$= a(b^{ch}-1)b^{c(x+h)} - a(b^{ch}-1)b^{cx}$$

$$= a(b^{ch}-1)b^{cx}(b^{ch}-1) = a(b^{ch}-1)^2 b^{cx}.$$

(iii) $\Delta \left[\frac{2^x}{(x+1)!} \right] = \frac{2^{x+1}}{(x+2)!} - \frac{2^x}{(x+1)!}$

taking the interval of differencing as unity

$$= \frac{2^x}{(x+1)!} \left[\frac{2}{x+2} - 1 \right]$$

$$= \frac{2^x}{(x+1)!} \left[\frac{2-x-2}{x+2} \right] = \frac{-x \cdot 2^x}{(x+2)!}.$$

(iv) $\Delta \cot 2^x$

$= \cot 2^{x+1} - \cot 2^x$, taking the interval of differencing as 1

$$= \frac{\cos 2^{x+1}}{\sin 2^{x+1}} - \frac{\cos 2^x}{\sin 2^x}$$

$$\begin{aligned}
 &= \frac{\cos 2^{x+1} \sin 2^x - \cos 2^x \sin 2^{x+1}}{\sin 2^{x+1} \cdot \sin 2^x} = \frac{\sin(2^x - 2^{x+1})}{\sin 2^x \cdot \sin 2^{x+1}} \\
 &= \frac{\sin \{2^x(1-2)\}}{\sin 2^x \cdot \sin 2^{x+1}} = \frac{-\sin 2^x}{\sin 2^x \cdot \sin 2^{x+1}} = -\operatorname{cosec} 2^{x+1}.
 \end{aligned}$$

$$\begin{aligned}
 (\text{v}) \quad \frac{\Delta^2}{E} x^3 &= \frac{(E-I)^2}{E} x^3 = \left[\frac{E^2 + I^2 - 2EI}{E} \right] x^3 \\
 &= [E + I^2 E^{-1} - 2I] x^3 = Ex^3 + I^2 E^{-1} x^3 - 2Ix^3 \\
 &= (x+1)^3 + (x-1)^3 - 2x^3 \\
 &= x^3 + 1 + 3x^2 + 3x + x^3 - 1 - 3x^2 + 3x - 2x^3 = 6x.
 \end{aligned}$$

$$\begin{aligned}
 (\text{vi}) \quad \frac{\Delta^2 x^3}{Ex^3} &= \frac{(E-I)^2 x^3}{Ex^3} = \frac{(E^2 + I^2 - 2E) x^3}{Ex^3} = \frac{E^2 x^3 + Ix^3 - 2Ex^3}{Ex^3} \\
 &= \frac{(x+2)^3 + x^3 - 2(x+1)^3}{(x+1)^3} = \frac{x^3 + 8 + 6x^2 + 12x + x^3 - 2(x+1)^3}{(x+1)^3} \\
 &= \frac{6(x+1)}{(x+1)^3} = \frac{6}{(x+1)^2}.
 \end{aligned}$$

$$\begin{aligned}
 (\text{vii}) \quad \Delta \sinh(a+bx) &= \sinh[a+(x+h)b] - \sinh[a+bx] \\
 &= 2 \cosh \frac{2a+2bx+bh}{2} \sinh \frac{bh}{2} \\
 &= 2 \cosh \left(a + \frac{b}{2} + bx \right) \sinh \frac{b}{2}, \text{ for } h=1.
 \end{aligned}$$

$$\begin{aligned}
 (\text{viii}) \quad \Delta \cosh(a+bx) &= \cosh[a+b(x+h)] - \cosh[a+bx] \\
 &= 2 \sinh \frac{2a+2bx+bh}{2} \sinh \frac{bh}{2} \\
 &= 2 \sinh \left(a + \frac{b}{2} + bx \right) \sinh \frac{b}{2}, \text{ for } h=1.
 \end{aligned}$$

Ex. 3. Evaluate

$$\begin{aligned}
 (\text{i}) \quad \Delta^3 [ax^3 + bx^2 + cx + d] &= \Delta^3 ax^3, \quad [\because \text{by } \S 1.9 \Delta^3 bx^2 = 0; \Delta^3 cx = 0 \text{ and } \Delta^3 d = 0] \\
 &= a \Delta^3 x^3 = a \cdot 3! h^3 = 6 ah^3.
 \end{aligned}$$

$$\begin{aligned}
 (\text{ii}) \quad \Delta \tan ax &= \tan a(x+1) - \tan ax = \frac{\sin a(x+1)}{\cos a(x+1)} - \frac{\sin ax}{\cos ax}, \text{ for } h=1 \\
 &= \frac{\sin a(x+1) \cos ax - \sin ax \cos a(x+1)}{\cos a(x+1) \cos ax} \\
 &= \frac{\sin [ax+a-ax]}{\cos ax \cos a(x+1)} = \frac{\sin a}{\cos ax \cos a(x+1)}.
 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & \Delta \tan^{-1} ax = \tan^{-1} a(x+1) - \tan^{-1} ax \\ &= \tan^{-1} \left[\frac{ax+a-ax}{1+(ax+a)ax} \right] = \tan^{-1} \frac{a}{1+a^2x^2+a^2x}. \end{aligned}$$

Ex. 4. Evaluate

(i) $\Delta^3 (1-x)(1-2x)(1-3x)$; (ii) $\Delta^n (e^{ax+b})$,
the interval of differencing being unity.

$$\begin{aligned} \text{Sol. (i)} \quad & \text{Here } f(x) = (1-x)(1-2x)(1-3x) \\ &= -6x^3 + 11x^2 - 6x + 1. \end{aligned}$$

Thus $f(x)$ is a polynomial in x of degree 3.

$$\begin{aligned} \therefore \Delta^3 f(x) &= -6\Delta^3 x^3 + 11\Delta^3 x^2 - 6\Delta^3 x + \Delta^3 1, \\ &= (-6)(3!) = -36, \text{ refer } \S 1.9. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \Delta (e^{ax+b}) &= f(x+1) - f(x), \text{ where } f(x) = e^{ax+b} \\ &= e^{a(x+1)+b} - e^{ax+b} = [e^{ax+b} (e^a - 1)], \end{aligned}$$

$$\begin{aligned} \text{and } \Delta^2 (e^{ax+b}) &= \Delta (\Delta e^{ax+b}) = \Delta [e^{ax+b} (e^a - 1)] \\ &= e^{a(x+1)+b} (e^a - 1) - e^{ax+b} (e^a - 1) \\ &= e^{ax+b} (e^a - 1)^2. \end{aligned}$$

Proceeding in a similar way, we get

$$\Delta^n (e^{ax+b}) = e^{ax+b} (e^a - 1)^n.$$

$$\text{Ex. 5. Prove that } e^x = \left(\frac{\Delta^2}{E} \right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x};$$

the interval of differencing being h .

Sol. We have

$$Ee^x = e^{x+h}.$$

Again $\Delta e^x = e^{x+h} - e^x = e^x (e^h - 1)$. Similarly $\Delta^2 e^x = e^x (e^h - 1)^2$.

$$\therefore \left(\frac{\Delta^2}{E} \right) e^x = (\Delta^2 E^{-1}) e^x = \Delta^2 e^{x-h} = e^{-h} \Delta^2 e^x = e^{-h} \cdot e^x (e^h - 1)^2.$$

$$\therefore \text{R.H.S.} = \left(\frac{\Delta^2}{E} \right) e^x \cdot \frac{(Ee^x)}{(\Delta^2 e^x)} = e^{-h} \cdot e^x (e^h - 1)^2 \cdot \frac{e^{x+h}}{e^x (e^h - 1)^2} = e^x.$$

Ex. 6. Evaluate $\Delta^2 (\cos 2x)$.

[Meerut 1984]

Sol. We have

$$\begin{aligned} \Delta^2 \cos 2x &= (E-1)^2 \cos 2x = (E^2 - 2E + 1) \cos 2x \\ &= E^2 \cos 2x - 2E \cos 2x + \cos 2x \\ &= E \cos 2(x+h) - 2 \cos 2(x+h) + \cos 2x \\ &= \cos \{2(x+2h)\} - 2 \cos \{2(x+h)\} + \cos 2x \\ &= \cos(2x+4h) - \cos(2x+2h) - \cos(2x+2h) + \cos 2x \\ &= 2 \sin(2x+3h) \sin(-h) + \{2 \sin(2x+h) \sin h\} \\ &= -2 \sin h [\sin(2x+3h) - \sin(2x+h)] \\ &= -2 \sin h [2 \cos(2x+2h) \sin h] \\ &= -4 \sin^2 h \cos(2x+2h). \end{aligned}$$

Ex. 7. Evaluate $\Delta^2 (3e^x)$.

Sol. We have $\Delta (3e^x) = 3\Delta (e^x) = 3(e^{x+h} - e^x) = 3e^x(e^h - 1)$.

$$\begin{aligned}\therefore \Delta^2 (e^x) &= \Delta \{3e^x(e^h - 1)\} = 3(e^h - 1)(\Delta e^x) \\ &= 3(e^h - 1)(e^{x+h} - e^x) = 3(e^h - 1)^2 e^x.\end{aligned}$$

Ex. 8. Evaluate $\Delta^2 e^x$.

[Meerut 1991]

Sol. Proceed as above.

Ex. 9. Evaluate $\Delta (x + \cos x)$; the interval of differencing is α .

Sol. We have

$$\begin{aligned}\Delta (x + \cos x) &= \Delta x + \Delta \cos x \\ &= \{(x + \alpha) - x\} + \{\cos(x + \alpha) - \cos x\} \\ &= \alpha + 2 \sin\left(\frac{2x + \alpha}{2}\right) \sin\left(-\frac{\alpha}{2}\right) \\ &= \alpha - 2 \sin\left(x + \frac{\alpha}{2}\right) \sin\frac{\alpha}{2}.\end{aligned}$$

Ex. 10. Evaluate $\Delta \tan^{-1} x$.

Sol. $\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x$

$$= \tan^{-1} \left\{ \frac{(x + h) - x}{1 + (x + h)x} \right\} = \tan^{-1} \left[\frac{h}{1 + hx + x^2} \right].$$

Ex. 11. Show that $\Delta \log f(x) = \log \left\{ 1 + \frac{\Delta f(x)}{f(x)} \right\}$.

[Meerut 1980]

Sol. We have

$$\begin{aligned}\Delta \log f(x) &= \log f(x + h) - \log f(x) \\ &= \log \left[\frac{f(x + h)}{f(x)} \right] = \log \left[\frac{Ef(x)}{f(x)} \right] \\ &= \log \left[\frac{(1 + \Delta) f(x)}{f(x)} \right] = \log \left[\frac{f(x) + \Delta f(x)}{f(x)} \right] \\ &= \log \left[1 + \frac{\Delta f(x)}{f(x)} \right].\end{aligned}$$

Ex. 12. Evaluate $\Delta^n [ax^n + bx^{n-1}]$.

Sol. We have $\Delta^n [ax^n + bx^{n-1}] = \Delta^n (ax^n) + b\Delta^n (x^{n-1})$
 $= a(n!) + b \cdot 0 = a(n!).$

Ex. 13. Evaluate

$$(i) \quad \Delta^3 \left[\frac{5x+12}{x^2+5x+6} \right]; \quad (ii) \quad \Delta^n \left(\frac{1}{x} \right).$$

$$\begin{aligned}\text{Sol.} \quad (i) \quad \Delta^3 \left[\frac{5x+12}{x^2+5x+6} \right] &= \Delta^3 \left[\frac{2(x+3)+3(x+2)}{(x+2)(x+3)} \right] \\ &= \Delta^3 \left[\frac{2}{x+2} + \frac{3}{x+3} \right] = \Delta \cdot \Delta \left[\frac{2}{x+2} + \frac{3}{x+3} \right].\end{aligned}$$

Ex. 14. Evaluate $\Delta^4 ae^x$.

Sol. $\Delta ae^x = a \Delta e^x = a [e^{x+h} - e^x]$
 $= ae^x [e^h - 1] = ae^x (e - 1)$, for $h=1$.

Now $\Delta^2 ae^x = \Delta [\Delta ae^x]$
 $= \Delta [ae^x(e-1)] = a (e-1) [e^{x+1} - e^x]$
 $= a(e-1) e^x (e-1) = a(e-1)^2 e^x$.

Proceeding like this, we get $\Delta^4 ae^x = a(e-1)^4 e^x$.