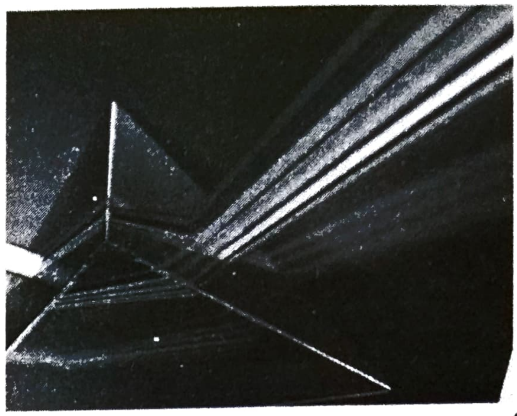

1 Chapter

SUPERPOSITION OF COLLINEAR HARMONIC OSCILLATIONS

1.1. PRINCIPLE OF SUPERPOSITION AND LINEARITY

The principle of superposition states that when two or more harmonic waves are simultaneously propagating in an elastic medium are super imposed then resultant displacement of any particle at any instant is equal to the vector sum of the displacements of that particle, due to individual (separate) waves at that instant. According to this principle each wave moves independently as if other waves were not present at all and their individual shapes and other characteristics are not changed due to the presence of another.

To explain this principle of superposition, let y_1 is the displacement of point x at time t. The wave is described by function $y_1\ (x,\ t)$. Hence this function must be the solution of differential equation of wave motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

That is

$$\frac{\partial^2 y_1\ (x,\ t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y_1(x,\ t)}{\partial t^2} \qquad \text{...(1.1)}$$

Similarly let y_2 be the displacement of the same point x at the same time t. This is also the solution of differential equation of wave motion

i.e.

$$\frac{\partial^2 y_2(x,\ t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y_2(x,\ t)}{\partial t^2} \qquad \text{...(1.2)}$$

Adding eqs. (1.1) and (1.2), we get

$$\frac{\partial^2 y_1(x,\ t)}{\partial x^2} + \frac{\partial^2 y_2\ (x,\ t)}{\partial x_2} = \frac{1}{v^2}\frac{\partial^2 y_1\ (x,\ t)}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 y_2(x,\ t)}{\partial t^2}$$

or $$\frac{\partial^2}{\partial x^2}[y_1(x,t)+y_2(x,t)] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}[y_1(x,t)+y_2(x,t)]$$

or $$\frac{\partial^2}{\partial x^2}y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t) \quad ...(1.3)$$

where $$y(x,t) = y_1(x,t) + y_2(x,t) \quad ...(1.4)$$

Hence eq. (1.3) shows that the sum of two wave functions described by eq. (1.4) also satisfy the differential equation of wave motion as satisfied by each separate wave function. Therefore $y(x,t)$ is the proper wave function to describe this placement of point x at time t. As it is clear from eq. (1.4) that the displacement y of the particle x at time t is equal to the sum of the displacements y_1 and y_2 of that point due to separate waves at the same time. Thus the principle of superposition is the consequence of differential equation of wave motion.

The differential equation of wave motion is linear and homogeneous. The differential equation is said to be linear if it contains the terms which depend only on the first powers of the variable and its derivatives. Similarly the equation is said to be homogeneous if it contains no terms independent of y. Thus *sum of two solutions is itself a solution.* It is also the statement of superposition principle. Thus superposition principle holds only for linear differential equations.

Thus the principle of superposition holds for those waves only whose equations of motion are linear. Thus it does not hold for shock waves created by explosions and water waves.

1.2. SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS HAVING EQUAL FREQUENCIES

Consider two simple harmonic oscillations of equal frequency but of different amplitudes and phase acting on a particle in y-direction.

Let y_1 and y_2 are displacements of two S.H.M's of same angular frequency ω are given by

$$y_1 = a\sin(\omega t + \phi_1) \quad ...(1.5)$$

and $$y_2 = b\sin(\omega t + \phi_2) \quad ...(1.6)$$

where a and b are the amplitudes and ϕ_1 and ϕ_2 are the phase respectively of two simple harmonic motion.

According to superposition principle the resultant of these two harmonic motion is given as

$$y = y_1 + y_2$$

or
$$\begin{aligned} y &= a\sin(\omega t + \phi_1) + b\sin(\omega t + \phi_2) \\ &= a[\sin\omega t\cos\phi_1 + \cos\omega t\sin\phi_1) + b[\sin\omega t\cos\phi_2 + \cos\omega t\sin\phi_2] \\ &= a\sin\omega t\cos\phi_1 + a\cos\omega t\sin\phi_1 + b\sin\omega t\cos\phi_2 + b\cos\omega t\sin\phi_2 \end{aligned}$$

$\therefore$ $$y = (a\cos\phi_1 + b\cos\phi_2)\sin\omega t + (a\sin\phi_1 + b\sin\phi_2)\cos\omega t \quad ...(1.7)$$

Put $$A\cos\theta = a\cos\phi_1 + b\cos\phi_2 \quad ...(1.8)$$

and $$A\sin\theta = a\sin\phi_1 + b\sin\phi_2 \quad ...(1.9)$$

in eq. (1.7), we get

$$y = A\sin\omega t\cos\theta + A\cos\omega t\sin\theta$$

or $$y = A(\sin\omega t\cos\theta + \cos\omega t\sin\theta)$$

or $$y = A\sin(\omega t + \theta) \quad ...(1.10)$$

Squaring and adding eqs. (1.8) and (1.9), we get

$$A^2 \sin^2 \theta + A^2 \cos^2 \theta = (a \sin \phi_1 + b \sin \theta_2)^2 + (a \cos \phi_1 + b \cos \phi_2)^2$$

$$A^2 (\sin^2 \theta + \cos^2 \theta) = a^2 \sin^2 \phi_1 + b^2 \sin^2 \phi_2 + 2ab \sin \phi_1 \sin \phi_2 + a^2 \cos^2 \phi_1 + b^2 \cos^2 \phi_2 + 2ab \cos \phi_1 \cos \phi_2$$

or
$$A^2 (1) = (a^2 \sin^2 \phi_1 + a^2 \cos^2 \phi_1) + (b^2 \sin^2 \phi_2 + b^2 \cos^2 \phi_2) + 2ab (\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2)$$

of
$$A^2 = a^2 (\sin^2 \phi_1 + \cos^2 \phi_1) + b^2 (\sin^2 \phi_2 + \cos^2 \phi_2) + 2ab \cos (\phi_1 - \phi_2)$$

or
$$A^2 = a^2 (1) + b^2 (1) + 2ab \cos (\phi_1 - \phi_2)$$

or
$$A^2 = a^2 + b^2 + 2ab \cos (\phi_1 - \phi_2) \qquad ...(1.11)$$

Dividing eq. (1.9) by eq. (1.8), we get

$$\frac{A \sin \theta}{A \cos \theta} = \frac{a \sin \phi_1 + b \sin \phi_2}{a \cos \phi_1 + b \cos \phi_2}$$

or
$$\tan \theta = \frac{a \sin \phi_1 + b \sin \phi_2}{a \cos \phi_1 + b \cos \phi_2} \qquad ...(1.12)$$

From eqs. (1.10), (1.11) and (1.12) shows that the resultant effect of two collinear S.H.M.'s of equal frequencies but having different amplitudes and phases is also a simple harmonic motion.

Special Cases

Let us consider the following special cases.

Case Ist when $\phi_1 - \phi_2 = 2n\pi$

where $n = 0, 1, 2, 3, ...$ etc. *i.e.* phase difference is even multiple of π

Then eq. (1.11) becomes

$$A^2 = a^2 + b^2 + 2ab \cos (2n\pi)$$

$$= a^2 + b^2 + 2ab (1) \qquad (\because \cos 2n\pi = 1)$$

$$= a^2 + b^2 + 2ab$$

$\therefore$
$$A^2 = (a + b)^2$$

or
$$A = a + b \qquad ...(1.13)$$

Case IInd. When $\phi_1 - \phi_2 = (2n + 1)\pi$

When $n = 0, 1, 2, 3, ...$ then $\cos (\phi_1 - \phi_2) = \cos (2n + 1) \pi = -1$

Then eq. (1.11) becomes

$$A^2 = a^2 + b^2 + 2ab (-1)$$

or
$$A^2 = a^2 + b^2 - 2ab$$

or
$$A^2 = (a - b)^2$$

or
$$A = a - b \qquad ...(1.14)$$

If $a = b$, then $A = 0$. The particle will remain at rest

Case IIIrd. If $a = b$ and $\phi_1 \neq \phi_2$

Then eq. (1.11) becomes
$$A^2 = a^2 + a^2 + 2a^2 \cos (\phi_1 - \phi_2)$$

$$= 2a^2 + 2a^2 \cos (\phi_1 - \phi_2) = 2a^2 [1 + \cos (\phi_1 - \phi_2)]$$

$$= 2a^2 \left[1 + 2 \cos^2 \frac{(\phi_1 - \phi_2)}{2}\right]$$

$$A^2 = 4a^2 \cos^2 \frac{(\phi_1 - \phi_2)}{2}$$

or
$$A = 2a \cos \frac{1}{2} (\phi_1 - \phi_2) \quad ...(1.15)$$

and eq. (1.12) becomes

$$\tan \theta = \frac{a \sin \phi_1 + a \sin \phi_2}{a \cos \phi_1 + a \cos \phi_2}$$

$$= \frac{a (\sin \phi_1 + \sin \phi_2)}{a (\cos \phi_1 + \cos \phi_2)} = \frac{\sin \phi_1 + \sin \phi_2}{\cos \phi_1 + \cos \phi_2}$$

$$\tan \theta = \tan \frac{1}{2} (\phi_1 + \phi_2)$$

or
$$\theta = \frac{1}{2} (\phi_1 + \phi_2) \quad ...(1.16)$$

The amplitude will be maximum A = $2a$ when $\cos \frac{1}{2} (\phi_1 - \phi_2) = 1$ or $\phi_1 - \phi_2 = 2n\pi$ where n is an integer. This is the case when component variations are in phase.

The amplitude will be zero when

$$\cos \frac{1}{2} (\phi_1 - \phi_2) = 0 \text{ or } \phi_1 - \phi_2 = (2n + 1)\ \pi,$$

this is the case when component variations are in opposite phase.

Graphical representation of resultant amplitude of superposition of two collinear S.H.Ms of equal frequency but of different amplitudes and phases are shown in Fig. (1.1).

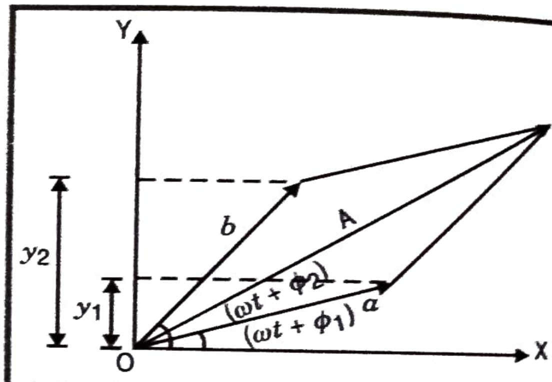

Fig. 1.1.

1.3. SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS WITH DIFFERENT FREQUENCIES (BEATS)

When two waves of nearly same frequency and having a constant initial phase difference propagate simultaneously along the same straight line and in the same direction, they superimpose and the intensity of sound at a point increases and decreases alternatively. This phenomena of increasing and decreasing of sound is called beats.

Let two sound waves of nearly same frequencies n_1 and n_2 (*i.e.* $n_1 > n_2$) travelling in the same direction along the same straight line pass through a point simultaneously. Let the two S.M.O. is are of nearly same angular frequencies ω_1 and ω_2 and having same amplitude a.

The wave equations of two S.H.M.s at any time t are

$$y_1 = a \sin \omega_1 t = a \sin 2\pi n_1 t \quad ...(1.17)$$

and
$$y_2 = a \sin \omega_2 t = a \sin 2\pi n_2 t \quad ...(1.18)$$

According to principle of superposition of waves the resultant displacement at time t is

$$y = y_1 + y_2$$

or
$$y = a \sin 2\pi n_1 t + a \sin 2\pi n_2 t$$

or
$$y = a (\sin 2\pi n_1 t + \sin 2\pi n_2 t) \quad ...(1.19)$$

According to formula

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Eq. (1.19) becomes

$$y = 2a\sin\frac{2\pi(n+n_2)t}{2}\cos\frac{2\pi(n_1-n_2)t}{2} \quad ...(1.20)$$

Let us take $\frac{n_1+n_2}{2} = n$

Then eq. (1.20) becomes

$$y = 2a\sin 2\pi nt \cos\pi(n_1 - n_2)t \quad ...(1.21)$$

Put

$$A = 2a\cos\pi(n_1 - n_2)t \quad ...(1.22)$$

in eq. (21), we get

$$y = A\sin 2\pi nt \quad ...(1.23)$$

where A is amplitude of the resultant wave

From eq. (1.23) it is clear that the resultant vibration is also simple harmonic vibration of amplitude A and frequency n.

Eq. (1.22) shows that amplitude 'A' of the resultant wave varies with time t.

Maxima

The intensity (of sound) at the observation point will be maximum if amplitude 'A' is maximum.

i.e. $\cos\pi(n_1 - n_2)t = 1$

or $\pi(n_1 - n_2)t = m\pi$

where $m = 0, 1, 2, 3, ...$

or

$$t = \frac{m}{(n_1 - n_2)} \quad ...(1.24)$$

Putting $m = 0, 1, 2, 3 ...$, we get

$$t = 0, \frac{1}{n_1-n_2}, \frac{2}{n_1-n_2}, \frac{3}{n_1-n_2} \quad ...(1.25)$$

At these instants the intensity is maximum

Time internal between two successive maxima $= \frac{1}{n_1 - n_2}$

$\therefore$ Frequency of maxima per second $= (n_1 - n_2)$.

Minima : For the intensity to be minimum at the observation point, amplitude 'A' should be minimum

i.e. $\cos\pi(n_1 - n_2)t = 0$

or $\pi(n_1 - n_2)t = (2m+1)\frac{\pi}{2}$ where $m = 0, 1, 2, 3.$

$$t = \frac{(2m+1)}{2(n_1-n_2)}$$

Putting $m = 0, 1, 2, 3 ...$

$$t = \frac{1}{2(n_1-n_2)}, \frac{3}{2(n_1-n_2)}, \frac{5}{2(n_1-n_2)} \quad ...(1.26)$$

At these instants the intensity is minimum

Time internal between two successive minima $= \dfrac{1}{n_1 - n_2}$

Eqs. (1.25) and (1.26) shows that maxima and minima occurs alternatively at equal intervals and sound becomes $(n_1 - n_2)$ times maximum and $(n_1 - n_2)$ times minimum in each second.

Thus number of beats per second = $(n_1 - n_2)$.

Formation of beats is shown in Fig. (1.2).

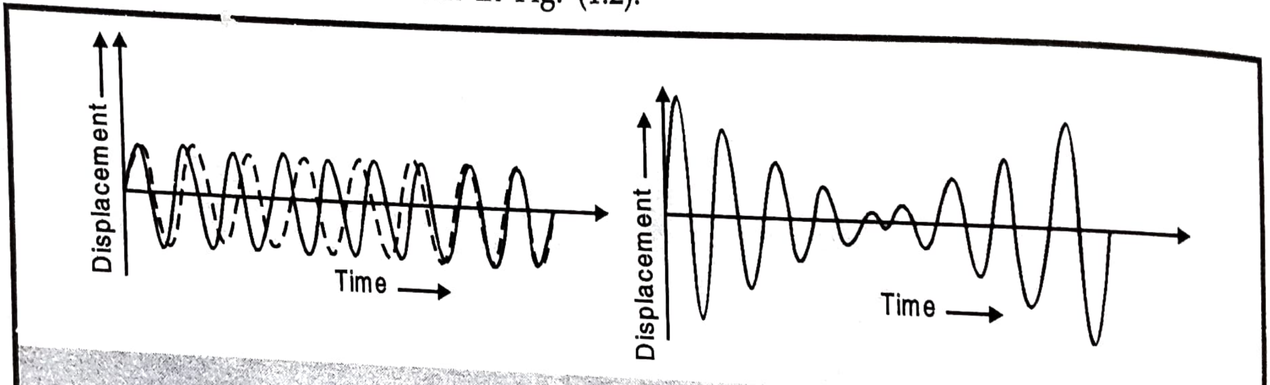

Fig. 1.2.