Fluid Pressure Under Gravity

2.1 : *Solid.* *A solid is that form of the matter that retains a definite shape and volume without any support. Pieces of iron, wood and ice etc, are examples of the solid state of matter.*

2.2 *Fluid.* *Fluid is that state of the matter which is capable of changing shape. It is capable of flowing also. Water and air etc. are examples of fluids.*

Further, fluids are also of two types, namely liquids and gases.

2.3 Liquid. A liquid like water or oil etc. is a fluid which is in-compressible *i.e.* has *a definite volume but has no definite shape. It is also called incompressible fluid*

2.4 Gases. A gas like air is that fluid which is capable of being compressed or is capable of being expanded almost indefinitely. *Thus a gas has no definite volume and no definite shape. It also called compressible fluid.*

2.5 : Perfect Fluid

A Perfect fluid is that ideal substance the particles of which yield at once to the slightest effort made to separate them from each other. Also the pressure of a perfect fluid, whether at rest or in motion, is only along the normal to the surface, there being no tangential force due to the fluid.

2.6 : Viscous Fluid

The particles of a viscous fluid exert resistance (friction) on each other or the motion of a body in a viscous fluid comes across the resistance due to the fluid.

a vessel with a horizontal base and vertical sides be filled with any fluid and a e be made in the side of the vessel and further let this hole be covered by a plate, ch exactly fits in it.

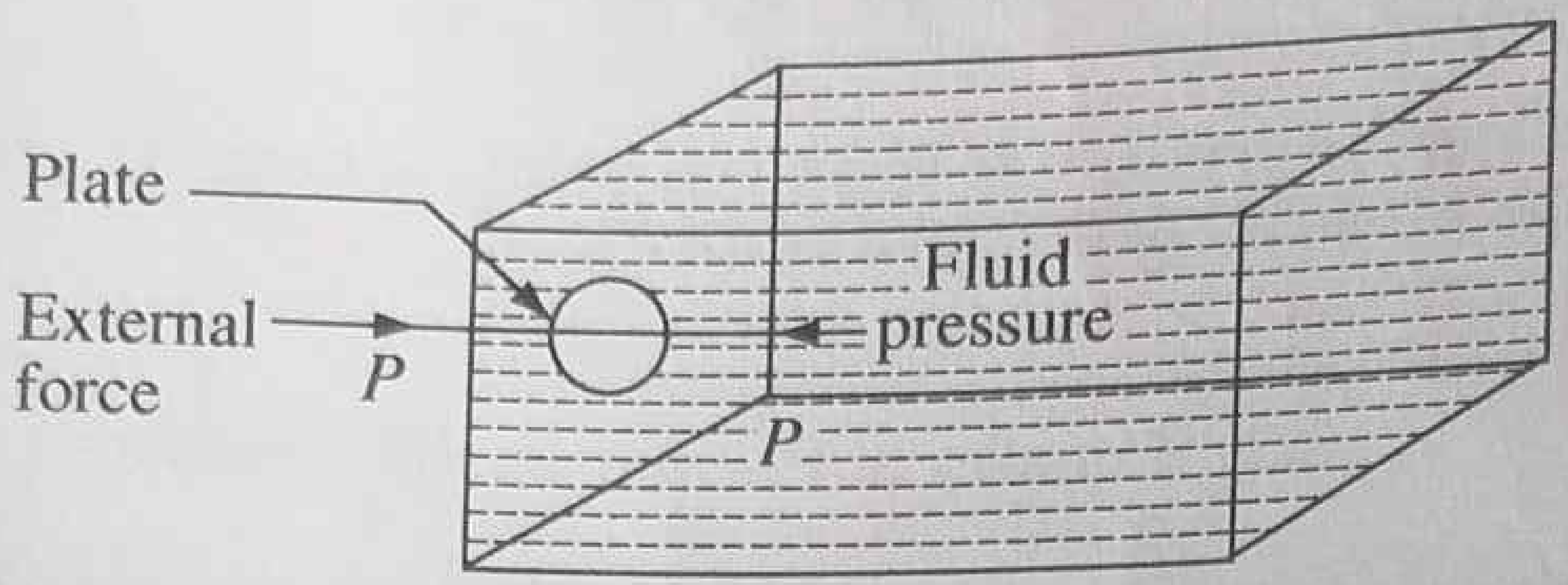

Fig. 3.1

ow it will be observed that the plate can be kept at rest only if some external is applied to it, which shows that the fluid in side the vessel is also exerting force on the plate. If P be the force necessary to keep the plate in equilibrium, P is called the *fluid pressure* or the *fluid thrust* on the whole area of the plate.

Uniform Pressure: If the fluid pressure is the same for each equal element of he area which can be on the horizontal base of the vessel or in other words he fluid pressure or any portion of the plane area is proportional to the area of that portion, the pressure on that area is said to be uniform.

On the other hand, if the pressure is not the same for different equal areas s it would be when the areas are taken on the vertical side of the vessel, the uid pressure is said to be varying or non-uniform.

- *Mean Pressure:* The mean pressure on a given plane area is the uniform pressure on it which will give the same resultant thrust on that area as the actual total pressure. Thus, if A be the area upon which a fluid thrust P is acting, then the mean pressure is $\frac{P}{A}$.

3.2 Pressure at a Point

The pressure at a point of a surface subjected to a fluid thrust is defined as the limit to which the ratio of the thrust on a very small area of the surface enclosing the point and the elementary area tends, as the elementary area is indefinitely diminished. In other words, the pressure at any point of an area is the limit of the mean pressure on a very small area enclosing that point.

Thus, if δP be the fluid pressure on a small area δA enclosing the given point in the fluid, the mean pressure per unit area

$$= \frac{\delta P}{\delta A}.$$

Now, let δA be diminished indefinitely, i.e., when $\delta A \to 0$, then

$$\lim_{\delta A \to 0} \frac{\delta P}{\delta A} = p$$

$$\text{or,} \quad p = \frac{dp}{dA}.$$

This p is called the *pressure* at the given point. Sometimes pressure at any point is also called the *intensity of pressure* at that point.

pressure.

Equality of Pressures in all Directions

2.9 : *The pressure at any point of a liquid (at rest) is same in all directions.*

(C.U. 04H; BNMU-14H)

Proof : Let O be any point inside the fluid. Take three mutually perpendicular directions OA, OB, OC and imagine a tetrahedron OABC of the fluid. Let P and P' be the mean pressures on the faces ABO and OBC respectively. Also let p be the pressure at the point O along a perpendicular to the plane ABC and p' be the pressure at O along OA, which is a line normal to the plane OBC. The result will be proved if we prove that $p = p'$.

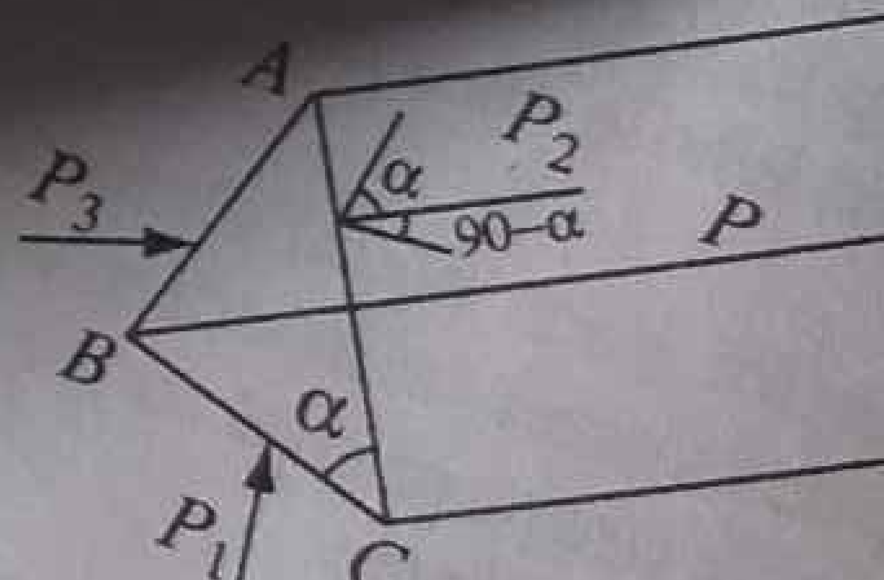

Fig. 3.2

Let W be the weight per unit v

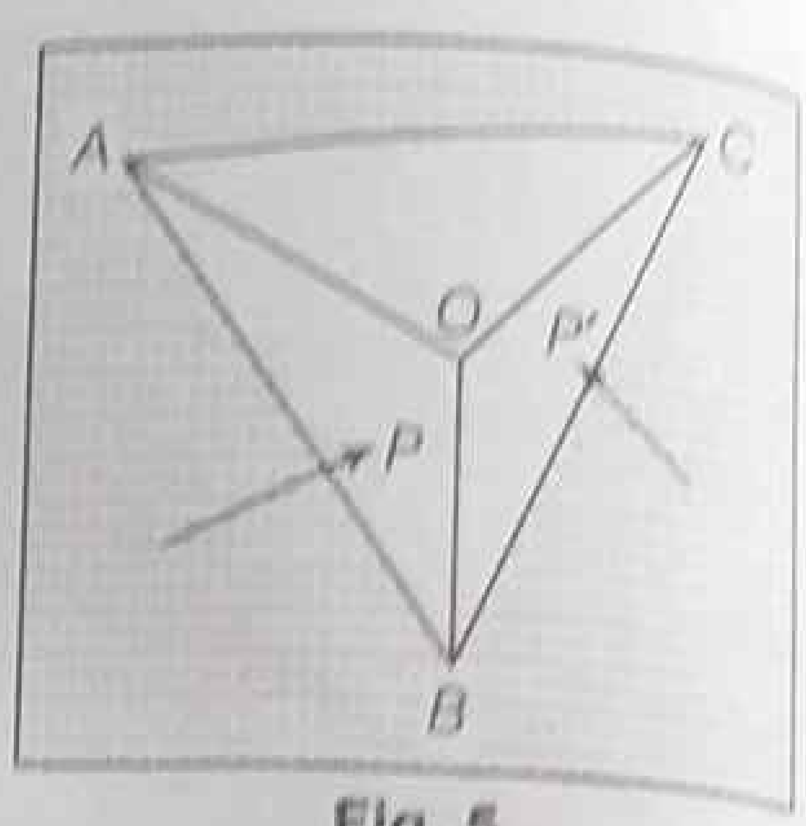

Fig. 5

Suppose that θ is the angle that the plane ABC makes with OBC or normal to ABC makes with OA, then

$$\Delta BOC = \text{projection of } ABC \text{ on } OBC$$

i.e.; $$\Delta OBC = \Delta ABC \cos\theta \qquad \ldots(1)$$

Since p and p' are the mean pressures on ΔABC and ΔOBC, the actual thrusts on them are $p \cdot \Delta ABC$ and $p' \cdot \Delta OBC$ respectively.

Thus the forces acting on this tetrahedron are:-

(i) The resultant of external forces. If gravity be the only force acting on the fluid, then the external force will be equal to the weight of the tetrahedron $= \frac{1}{2}\,\Delta OBC \times OA\rho g$.

(ii) Thrust $p \cdot \Delta ABC$ acting along the normal to ΔABC.

(iii) Thrust $p' \cdot \Delta OBC$ acting along OA.

(iv) The thrusts on the faces AOB and AOC acting along OC and OB respectively.

The tetrahedron is at rest under the action of the above forces; hence the algebraic sum of all the forces acting on it along any direction should vanish. If φ is the angle that OA makes with the vertical (direction of height), then resolving all the forces along OA, we get,

$$p' \cdot \Delta OBC - p \cdot \Delta ABC \cos\theta + \rho g \frac{1}{2} OA \cdot \Delta OBC \cos\varphi = 0$$

$$\Rightarrow \quad p' \cdot \Delta OBC - p \cdot \Delta OBC + \frac{1}{2} OA \cdot \Delta OBC \rho g \cos\varphi = 0 \qquad \text{(using (1))}$$

$$\Rightarrow \quad P' - p + \frac{1}{2} OA \cdot \rho g \cos\varphi = 0 \qquad \ldots(2)$$

Now move the plane ABC closer and closer to O, so that the tetrahedron diminishes indefinitely and OA ends to zero. Thus when $OA \to 0$, the tetrahedron reduces to a point.

$$\therefore \quad P \to p \quad \text{and} \quad P' \to p'$$

Thus, (2) gives $p' - p = 0$ in limit i.e. $p' = p$.

As the plane ABC, is an arbitrary plane, so its normal direction is also arbitrary along which p acts. This pressure p along any arbitrary direction is equal to pressure along OA. This proves the theorem for liquids at rest.

3.6 Pressure at Points in a Horizontal Plane

Theorem 3.6.1 *In a fluid, at rest under gravity, the pressure is the same at all points in the same horizontal plane.* [R.U. 80H, 94H, L.N.M.U. 93H]

Let A and B be any two points in fluid in the same horizontal plane. Join A and B. Then two cases arise: line AB may be wholly inside the fluid or it may not lie entirely in the fluid. We will consider the two cases separately.

Case I. *When the straight line AB lies wholly inside the fluid.*

With AB as axis, we construct a right circular cylinder of small cross-sectional area α. Let p_1 and p_2 be the pressures at A and B, respectively.

Since the ends of the cylinder are small, we can assume the pressure on them to be uniform and equal to the pressures at A and B, i.e., p_1 and p_2, respectively.

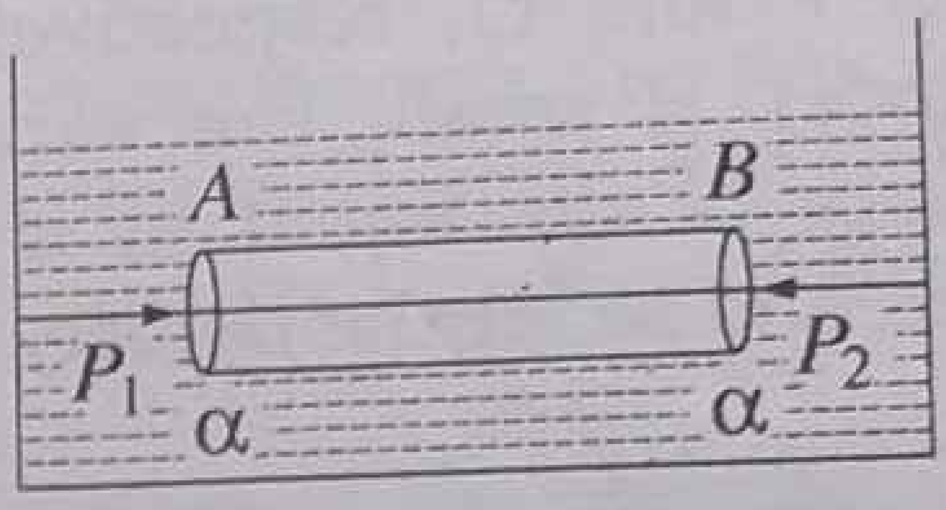

Fig. 3.5

Now the fluid in the cylinder is in equilibrium under the action of the following forces:

- The pressure $p_1\alpha$ on the end at A along AB.
- The pressure $p_2\alpha$ on the end at B along BA.
- The weight of the fluid in the cylinder acting vertically downwards.
- The pressure at each point on the curved surface of the cylinder, which are perpendicular to AB.

Hence, for equilibrium resolving all the forces along AB, we get

$$p_1\alpha - p_2\alpha = 0 \quad \text{or,} \quad p_1 = p_2,$$

i.e., Pressure at A = Pressure at B.

Case II. *When the straight line AB does not lie wholly inside the fluid.*

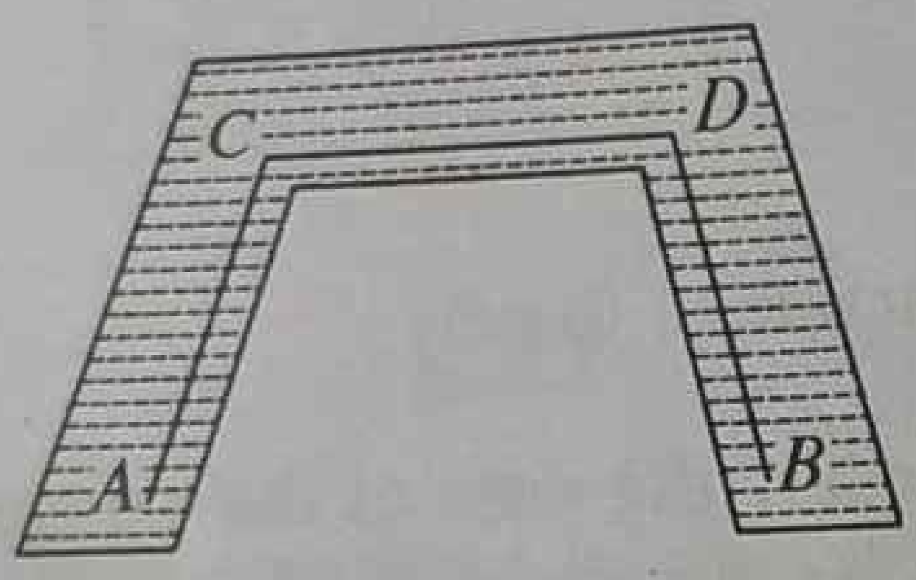

Fig. 3.6

Join A and B by a series of straight lines AC, CD, DB such that each of them lie wholly in the fluid. Then by case I

Pressure at A = Pressure at C

= Pressure at D

= Pressure at B.

Hence, the pressures at A and B are same.

segment doesn't lie wholly within the fluid.

Hence the theorem.

§ 2.12 : Find the pressure at a point of a liquid fluid

Or

Establish pressure equation for a Fluid at rest under gravity.

(TMBU-2006H, 2007H, 2009H, 11(H) B.N.M.U. 11H, 13H, 15H)

Solution

Let P and Q be two points in the same vertical line in a fluid of density ρ

Let z and $z + dz$ be the depth of P and Q below the fluid surface.

Then $PQ = dz$ which is very-very small.

Hence there will be no variation of density ρ and gravity g through the height PQ.

Let us construct an imaginary cylinder with axis PQ and cross-section α.

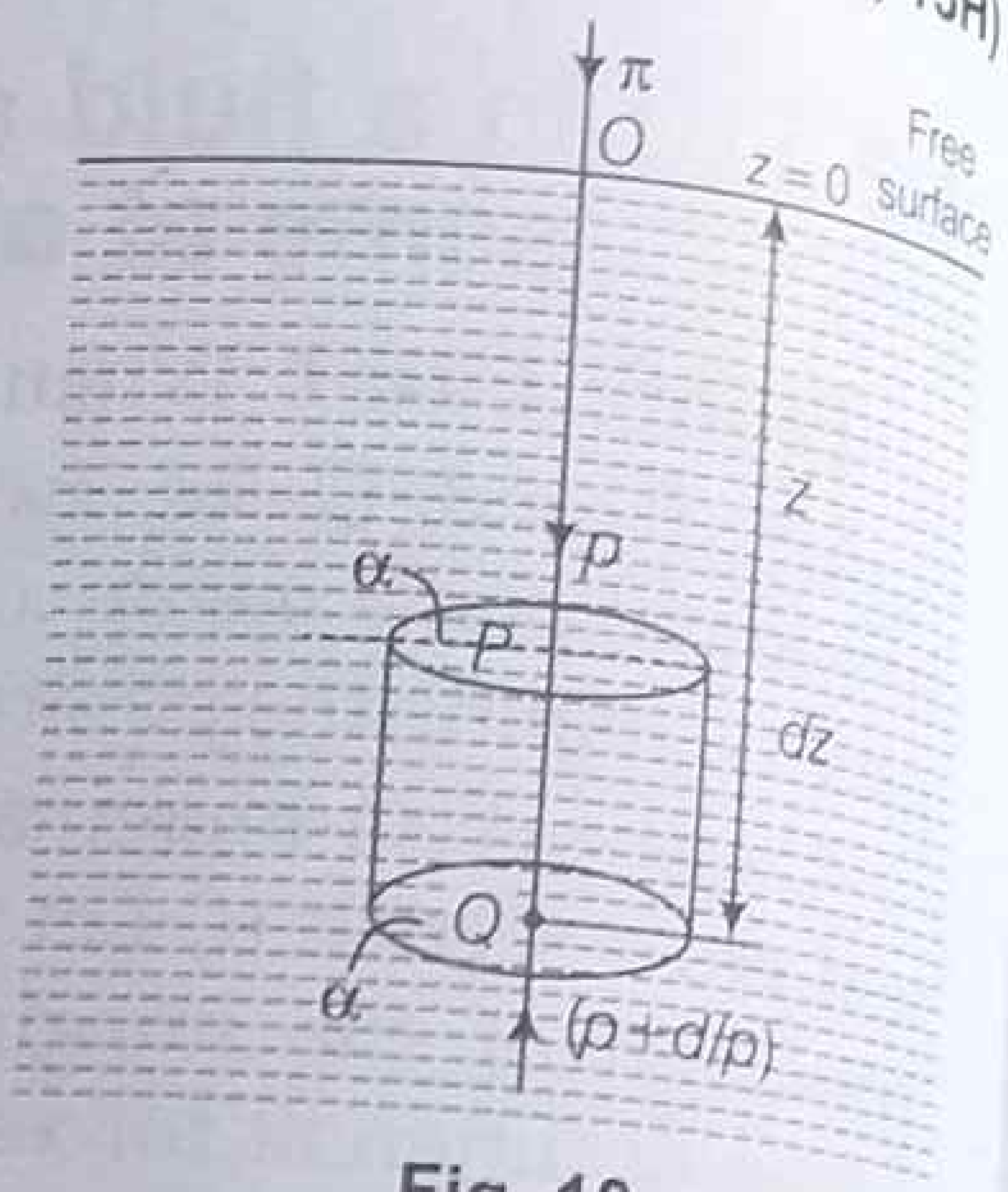

Fig. 10

at a depth h below the original surface of the liq...

uperimposed liquid is termed the *effective surface.*

in a homogeneous liquid is proportional to the depth

Let p be the fluid pressure per unit area at plane end at P along PQ and $(p+dp)$ be the fluid pressure per unit area at plane end Q along QP.

Since the water body is at rest under gravity, hence the imaginary cylinder is also in equilibrium position under the action of following forces :

(i) The thrust $p\alpha$ on the plane end at P along PQ.

(ii) The weight $\rho g\,\alpha\,dz$ of the liquid within the cylinder acting vertically downwards

(iii) Upward thrust $(p+dp)\,\alpha$ on the plane end at Q along QP.

(iv) The horizontal thrust on the curved surface of the cylinder.

Resolving all the forces horizontally and vertically we get,

$$p\alpha + \rho g\alpha \cdot dz = (p+dp)\alpha$$

$$\Rightarrow \quad dp = \rho g \cdot dz \qquad \ldots(1)$$

Equation (1) is the differential equation of pressure of a fluid of density ρ at depth z.

Case I When the fluid is homogeneous then ρ is constant.

Integrating (1), $$p = \rho g z + C, \qquad \ldots(2)$$

Where C is constant of integration.

Initially at the fluid surface, $z = 0$, $p = \Pi$ = atmospheric pressure, then from (2)

$$\Pi = \rho g \cdot 0 + C$$

$$\Rightarrow \quad C = \Pi$$

$$\therefore \quad p = \Pi + \rho g z \qquad \ldots(3)$$

Equation (3) is the required expression of pressure at depth z.

If we neglect the atmospheric pressure, then $\Pi = 0$

$\therefore$ From (3), we get

$$p = \rho g z$$

$$\Rightarrow \quad p \propto z \qquad \ldots(4)$$

$\therefore$ pressure is proportional to depth of fluid. **TMBU 2013(H)**

Case II When the fluid is heterogeneous then density ρ will be a function of z.

Let $p = f(z)$ then from eqn (1).

$$dp = f(z) g\, dz$$

$$\therefore \quad \int dp = \int f(z) \cdot g \cdot dz$$

$$\Rightarrow \quad p = g \cdot \int f(z)\, dz + C$$

This is the required expression of pressure for a heterogeneous fluid at depth z.

Corollary : Difference of pressures at two points varies as the difference of their depths. **(B.N.M.U. 12H)**

Proof :

Let p_1 and p_2 be the pressures of a fluid of density ρ at depths z_1 and z_2 respectively then

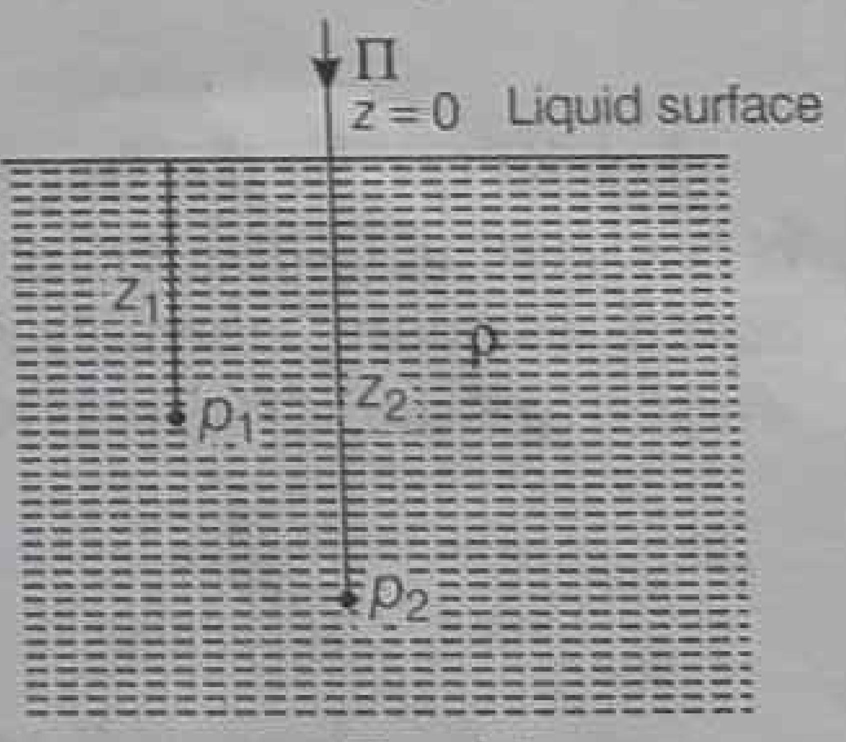

Fig. 11

$$p_1 = \Pi + \rho g z_1$$

$$p_2 = \Pi + \rho g z_2$$

Subtracting, $$p_2 - p_1 = \rho g\,(z_2 - z_1)$$

$$\Rightarrow \quad (p_2 - p_1) \propto (z_2 - z_1)$$

i.e. Difference of pressures varies as difference of depths.

Proved.

■ 2.13 : Free surface and effective surface of a liquid :

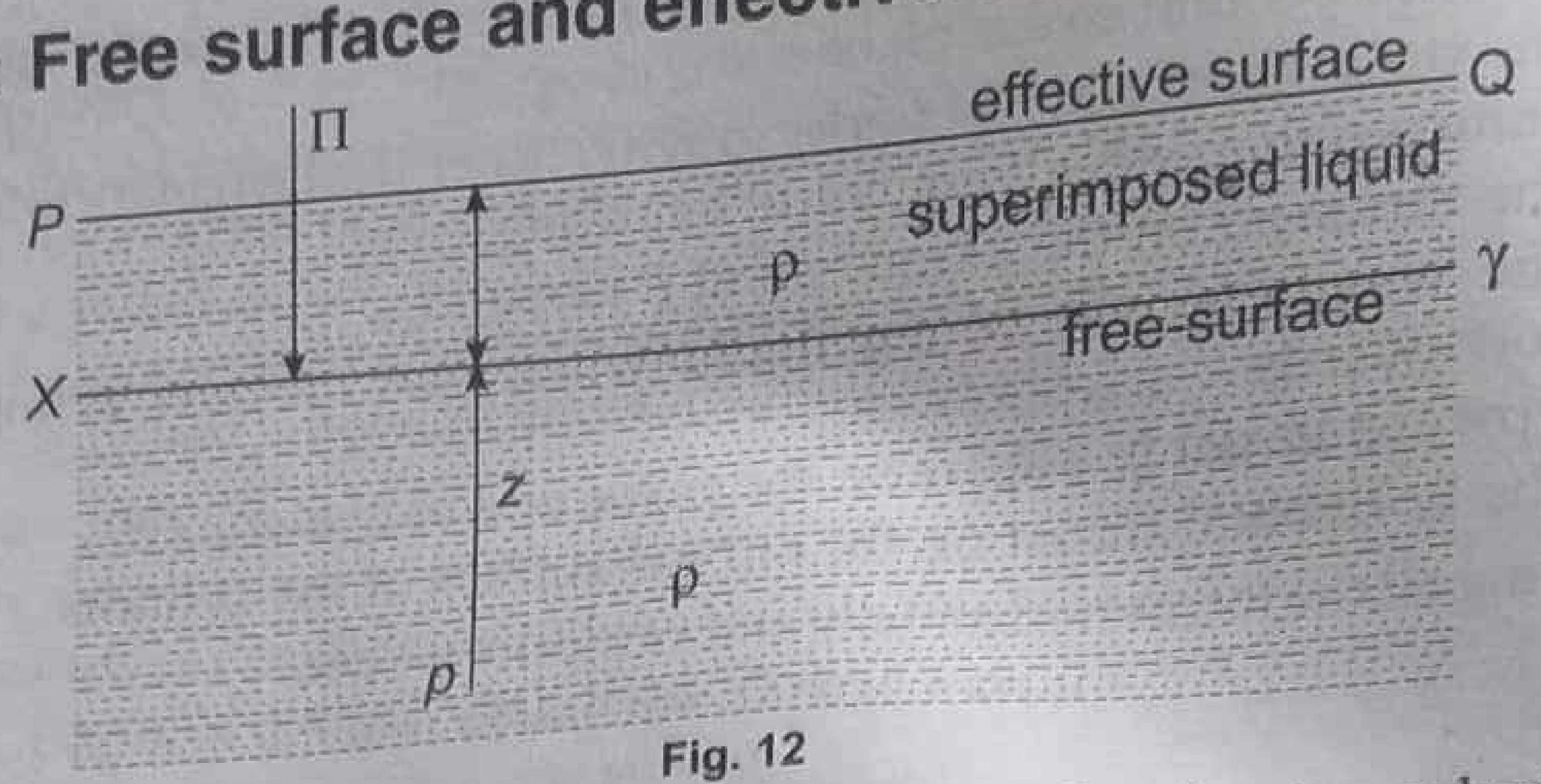

Fig. 12

Let XY be the free surface of the liquid of density ρ. Let the atasospheric pressure Π be equivalent to the pressure due to a stratum of the liquid of thickness h and density ρ, then $\Pi = \rho g h$.

Thus the total pressure at a point at depth z in the liquid $= \Pi + \rho g z$

i.e.,
$$p = \rho g h + \rho g z$$
$$= \rho g \,(h + z)$$

Now, if a stratum of liquid of thickness h and density ρ is superimposed on the free surface of the liquid, then the upper surface PQ of the superimposed liquid can be defined as the effective surface of the liquid.

Remark : (1) If the liquid is homogeneous, then ρ is constant and hence $p \propto (h + z)$ *i.e.,* the total pressure at any point in a homogeneous liquid is proportional to the depth of the point below the effective surface.

Remark : (2) As the liquid is incompressible, so we should not think about the free surface of the liquid that it may be lowered due to atmospheric pressure.

■ 2.14 : The free surface of a heavy homogeneous liquid at rest under gravity is horizontal.

[(TMBU-2008, 2010(H), 2012(H), 2014(H), 2016(H)]

Proof :

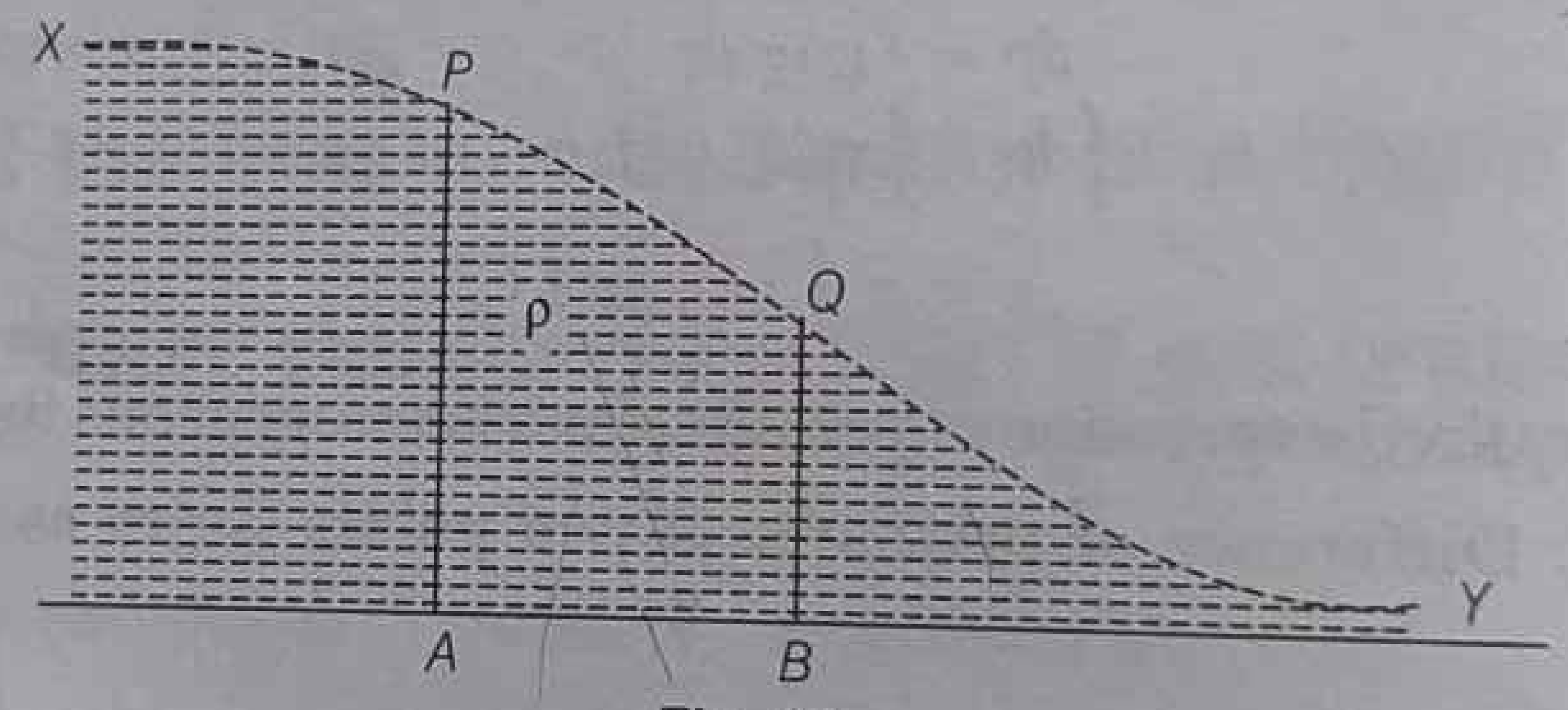

Fig. 13

Let $XPQY$ be the free surface of a liquid of density ρ. We have to prove that $XPQY$ is horizontal. Considering two points A and B on the same horizontal plane within the liquid. Drawing normals AP and BQ on the plane of A and B which intersect the free surface at P and Q respectively. Since A and B lie on the same horizontal plane, hence

pressure at A = pressure at B.

i.e.,
$$\rho g \cdot AP = \rho g \cdot BQ$$
i.e.,
$$AP = BQ$$

⇒ i.e., plane of P and Q is horizontal.

⇒ The free surface of a liquid is horizontal.

Remarks : From the definition, effective surface is also horizontal.

2.15 : In the same horizontal plane, the densities at two points in a liquid at rest under gravity are equal.

Proof : Let ρ_1 and ρ_2 be the densities of a liquid at two points A and B lying on the same horizontal line in the liquid at rest under gravity.

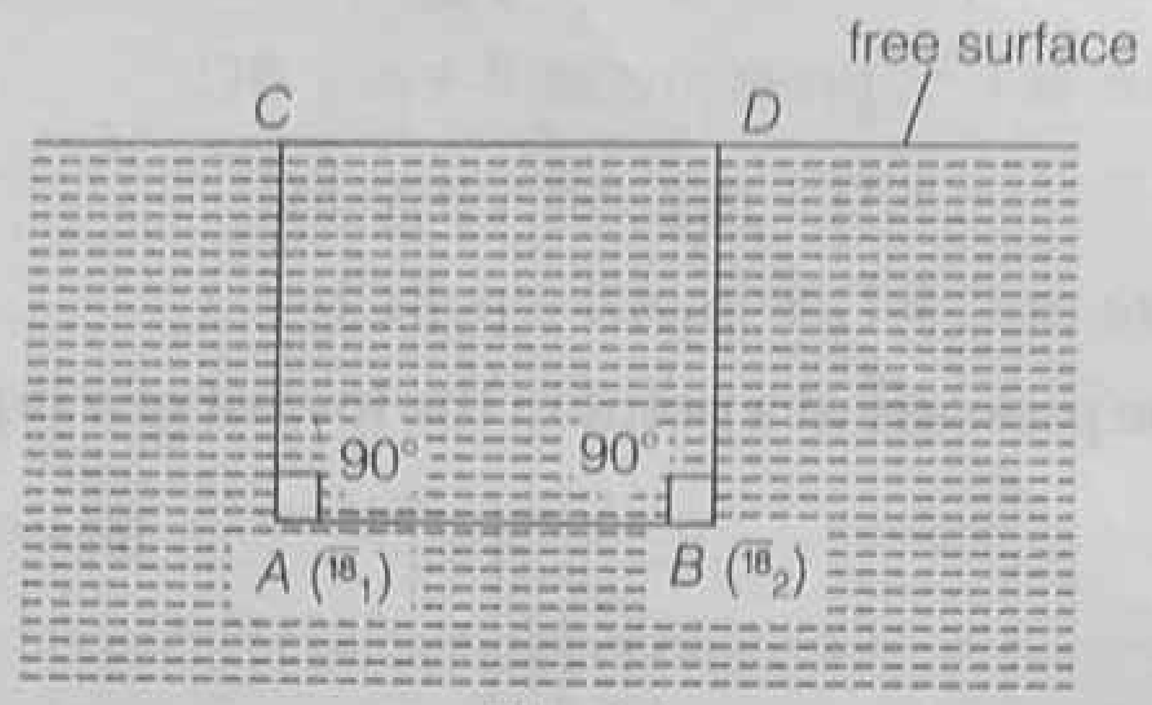

Fig. 14

Drawing normals AC and BD at A and B respectively, which meet the free surface at C and D respectively.

We know that free surface of any liquid is always horizontal, so CD is horizontal.

$$\therefore \quad AC = BD$$

As A and B lie on the same horizontal plane, so,

$$\text{pressure at } A = \text{pressure at } B$$

$$\Rightarrow \quad \rho_1 \cdot g \cdot AC = \rho_2 \cdot g \, BD$$

$$\Rightarrow \quad \rho_1 = \rho_2 \qquad (\because AC = BD)$$

$\therefore$ Density of any liquid at the same horizontal plane is always same.

§ 2.16 : To find the pressure at any given depth in the lower of two given heavy homogeneous liquids which don't mix.

Proof : Let XY be the effective surface of the upper liquid of density ρ_1 and PQ be the surface of separation of two liquids.

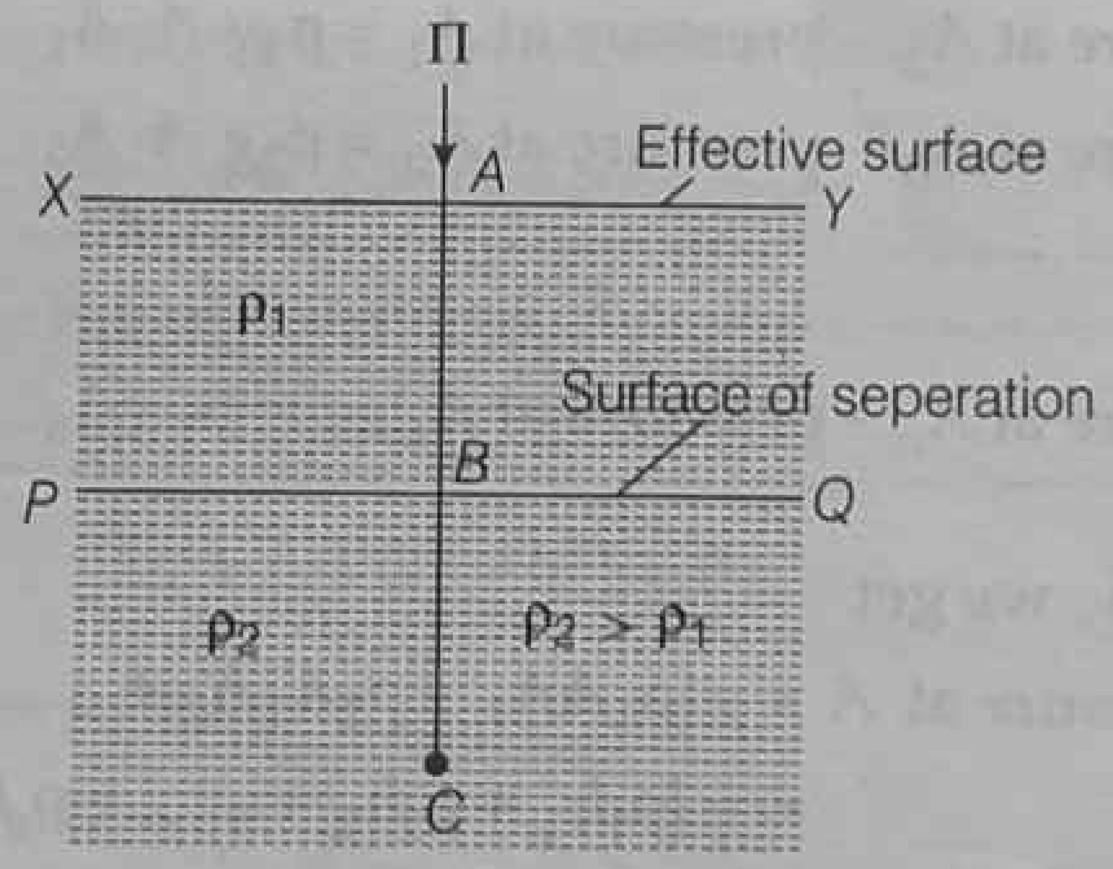

Fig. 15

Let ρ_2 be the density of lower liquid, where $\rho_2 > \rho_1$.

Taking a point A on the effective surface XY and drawing a vertical line through A which meets the surface of seperation at B.

Let C be a point in the lower liquid just vertically below B.

Thus A, B, C all lie on the same vertical line.

Let Π be the atmospheric pressure at A, then from the properties of liquids, pressure at B – pressure at $A = \rho_1 g \cdot AB$

$\therefore$ pressure at B = pressure at $A + \rho_1 g AB$

$= \Pi + \rho_1 g AB$ ...(1)

Again, pressure at C – pressure at $B = \rho_2 g \cdot BC$

$\therefore$ pressure at C = pressure at $B + \rho_2 g\ BC$

$= \Pi + \rho_1 g\ AB + \rho_2 g\ BC$ {using (1)}

i.e., pressure at $C = \Pi + (\rho_1 AB + \rho_2 BC) g$

Corollary : To find the pressure at a point in the lowest of n liquids which don't mix.

Proof :

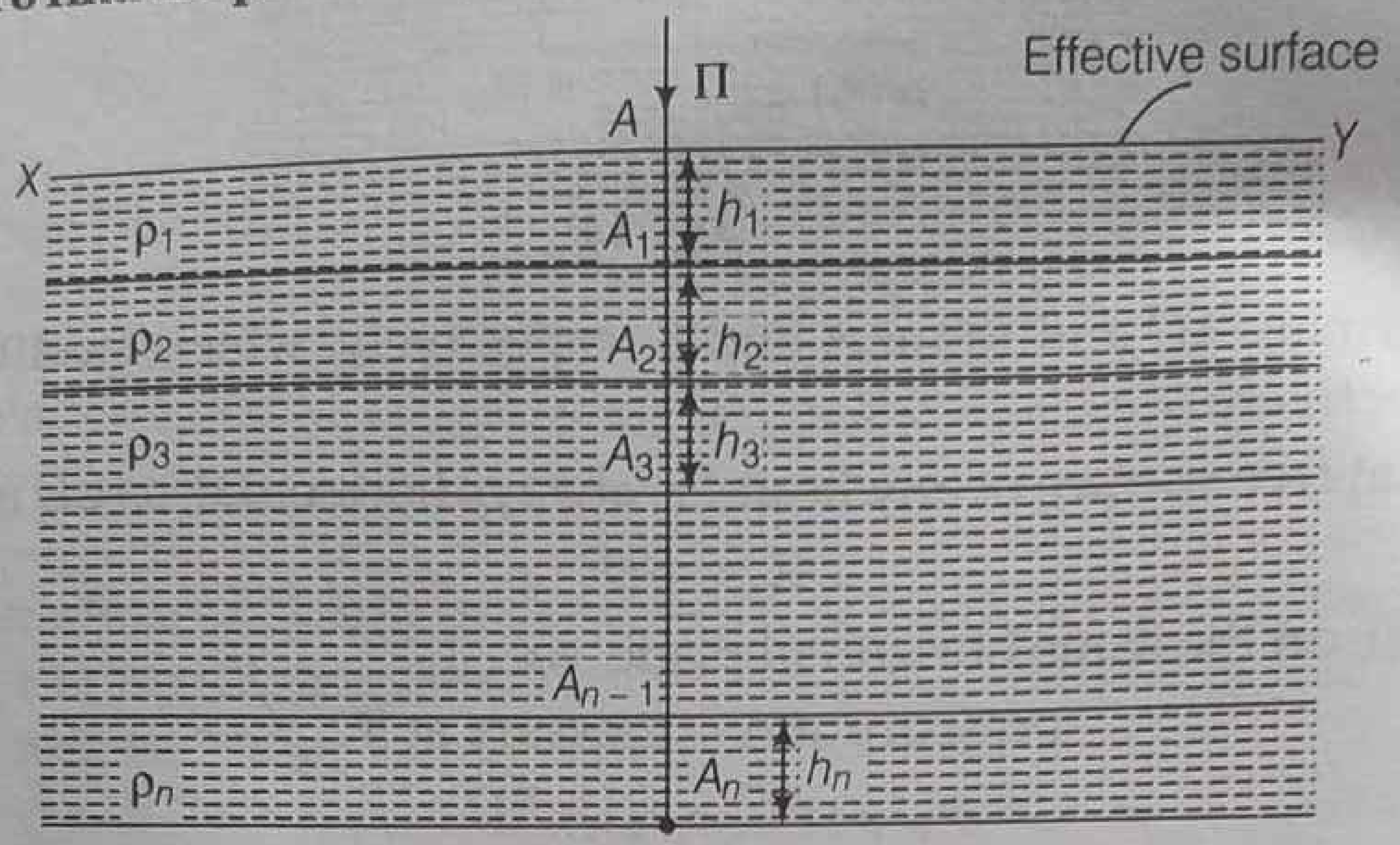

Fig. 16

Let A be a point on the effective surface of the uppermost liquid.

Now drawing a vertical line through A which meets the surfaces of seperation of n liquids at A_1, A_2, A_3 .... A_{n-1}.

Let A_n be a point in lowest liquid lying in the vertical line $A_1, A_2, A_3, \ldots, A_{n-1}$.

Let Π be the atmospheric pressure at A, then

pressure at A_1 – pressure at $A = \rho_1 g\ AA_1$

pressure at A_2 – pressure at $A_1 = \rho_2 g\ A_1A_2$

pressure at A_3 – pressure at $A_3 = \rho_3 g\ A_2A_3$

....

....

pressure at A_n – pressure at $A_{n-1} = \rho_n\ g A_{n-1}\ A_n$

Adding, we get

pressure at A_n – pressure at $A = g\,[\rho_1 AA_1 + \rho_2 A_1A_2 + \ldots\ldots\ldots + \rho_n A_{n-1}A_n]$

$= g\,[\rho_1 h_1 + \rho_2 h_2 + \ldots\ldots + \rho_n h_n]$

$= g \sum_{r=1}^{n} p_r h_r$

$\therefore$ pressure at A_n = pressure at $A + g \sum_{r=1}^{n} \rho_r\, h_r$

i.e., pressure at $A_n = \Pi + g \sum_{r=1}^{n} \rho_r h_r$