...fluids and the

pressure of fluids revolving about a vertical axis of the container

6.1 : Force Field

If in any domain of space each particle is subjected to a force which is proportional to the mass acted upon, then the domain is called field of force.

The force per unit mass is called the intensity of the field.

Equation of Pressure

§ 6.2 : *A mass of fluid is at rest under the action of given forces ; to obtain the equation of the pressure at any point of the fluid.*

Let (x, y, z) be the co-ordinates of a point P in the fluid, referred to rectangular axes. Take a point Q very nearer to P, such that PQ is parallel to the axis of x. Let the co-ordinates of Q be $(x + \delta x, y, z)$.

Construct a small cylinder (or prism) about axis PQ having its plane ends perpendicular to PQ.

Let α be the area of either plane end of the cylinder. Also let p be the pressure at P and $p + \delta p$, the pressure at Q. Therefore thrust at the plane end at P is $p\alpha$ and that at Q is $(p + \delta p)\, \alpha$.

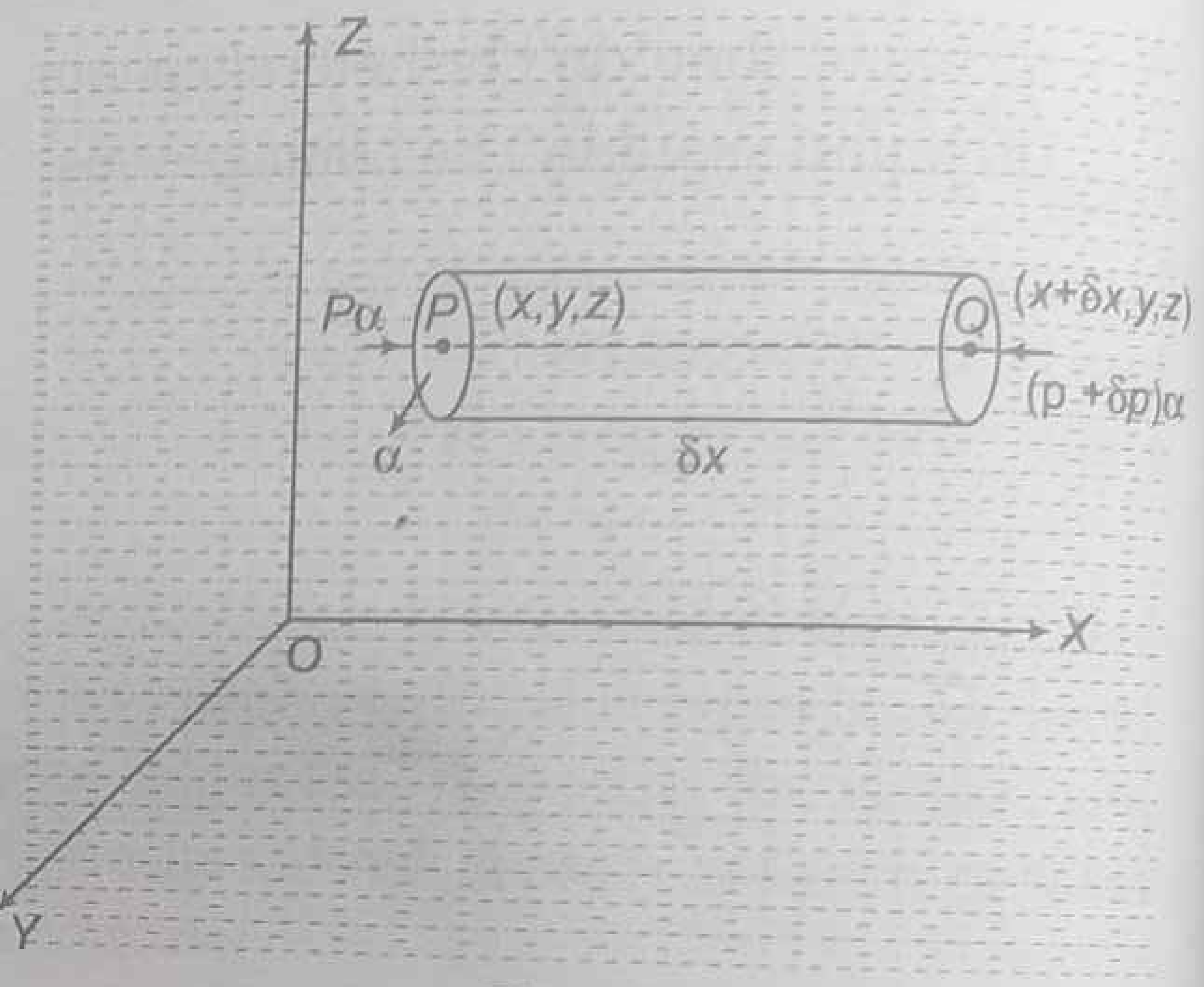

Fig. 160

Now let ρ be the mean density of the cylinder PQ; then mass of the cylinder $= \rho\alpha\, \delta x$.

If X, Y, Z be the components of the given force per unit mass, then the force on the cylinder parallel to the x-axis is $X\rho\alpha\, \delta x$.

Thus the cylinder is in equilibrium under the forces (i) $p\alpha$ along PQ, (ii) $X\alpha\rho\, \delta x$ along PQ, and (iii) $(p + \delta p)\, \alpha$ along QP as shown in the figure; thus for equilibrium of the cylinder, we have

$$(p + \delta p)\, \alpha = p\alpha + X\alpha\rho\, \delta x$$

or $\quad \delta p = \rho X\, \delta x \quad$ or $\quad \dfrac{\delta p}{\delta x} = \rho X.$

Proceeding to limit as δx (and therefore δp) tends to zero,

we get, $$\lim_{\delta x \to 0} \frac{\delta p}{\delta x} = \rho X \Rightarrow \frac{\partial p}{\partial x} = \rho X. \qquad \ldots(1)$$

Similarly, we have $$\frac{\partial p}{\partial y} = \rho Y \text{ and } \frac{\partial p}{\partial z} = \rho Z$$

But $$p = p\,(x, y, z) \qquad \ldots(2)$$

So $$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz$$

$$= \rho X\, dx + \rho Y\, dy + \rho Z\, dz. \qquad \text{[using (1)]}$$

Thus $$dp = \rho\,(X\, dx + Y\, dy + Z\, dz). \qquad \ldots(3)$$

This is the differential equation of pressure at any point $P(x, y, z)$.

§ 6.3 : Pressure when a give force acts in any given direction

Let F be the force acting along the given direction PQ, where $PQ = \delta s$ is small. Construct a cylinder about PQ as axis and plane ends perpendicular to PQ. If p is the pressure at the point P and α, the area of cross-section perpendicular to PQ and $p + \delta p$ pressure at Q, then the cylinder is in equilibrium under the following forces :

(i) $p\alpha$ along PQ,
(ii) $(p + \delta p)\,\alpha$ along QP,
(iii) $F \rho \alpha\, \delta s$, where ρ is the density of fluid, along PQ.

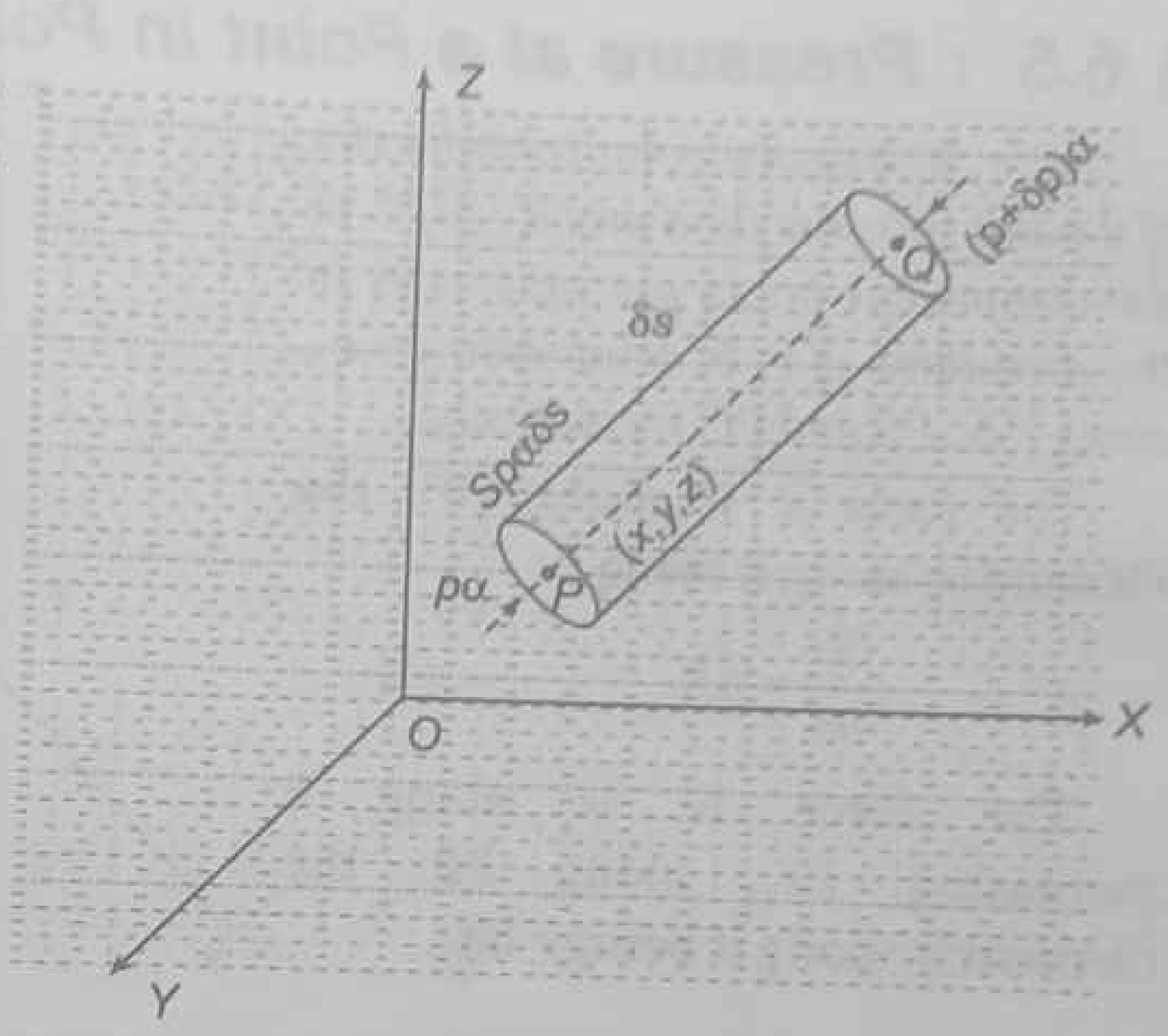

Fig.161

Thus for equilibrium of the cylinder, we have

$$(p + \delta p)\,\alpha = p\alpha + F \rho \alpha\, \delta s$$

or $$\delta p = F \rho\, \delta s.$$

Proceeding to limits, we get $dp = \rho F\, ds$ or $\dfrac{dp}{ds} = \rho F.$

This proves **that the rate of increase of the pressure in any direction is equal to the product of the density and the resolved part of the force in that direction.**

§ 6.4 : Pressure at a Point in Cylindrical Co-ordinates

Let (r, ϕ, z) be the cylindrical co-ordinates of a point in the fluid. If R, T and Z be the component forces per unit mass in the directions of r, ϕ, z increasing respectively, we have from § 6.2, $\dfrac{\partial p}{\partial s} = \rho R$, where R is the force in the direction of s increasing by an amount δs. In cylindrical co-ordinates the element δs in the direction of r increasing is δr.

$$\therefore \quad \frac{\partial p}{\partial r} = \rho R. \qquad \ldots(1)$$

Also the element δs in the direction of ϕ increasing is $r\ \delta\phi$.

$$\therefore \quad \frac{\partial p}{r\,\partial\phi} = \rho T \quad \text{or} \quad \frac{\partial p}{\partial\phi} = \rho r\, T \qquad \ldots(2)$$

and the element δs in the direction of z increasing in δz.

$$\therefore \quad \frac{\partial p}{\partial z} = \rho Z. \qquad \ldots(3)$$

Now since p is a function of r, ϕ and z, we have

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial\phi} d\phi + \frac{\partial p}{\partial z} dz$$

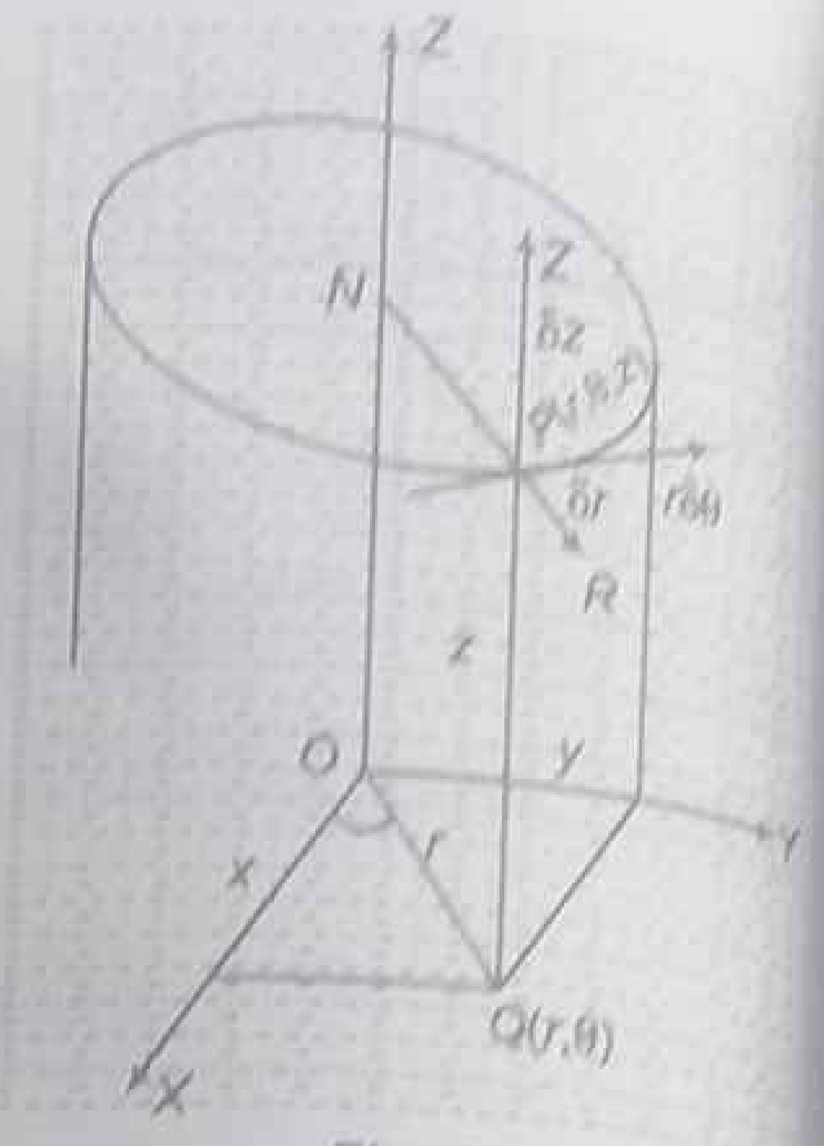

Fig. 162

$\therefore$ From (1), (2) & (3), we get,

$$dp = \rho\,[P\,dr + rT\,d\phi + Z\,dz] \qquad \ldots(4)$$

This is the required pressure equation in cylindrical coordinates.

§ 6.5 : Pressure at a Point in Polar Co-ordinates

Let r, θ, ϕ be the polar co-ordinates of a point P in the fluid and R, N, T be the component forces per unit mass in the directions of r, θ, ϕ increasing i.e., along PA, PB and PC respectively.

The force R is along OP, the element δs along it being δr.

$$\therefore \quad \frac{\partial p}{\partial r} = \rho R. \qquad \ldots(1)$$

The force N is along θ increasing, *i.e.*, along PB, the element δs along it being $r\ \delta\theta$.

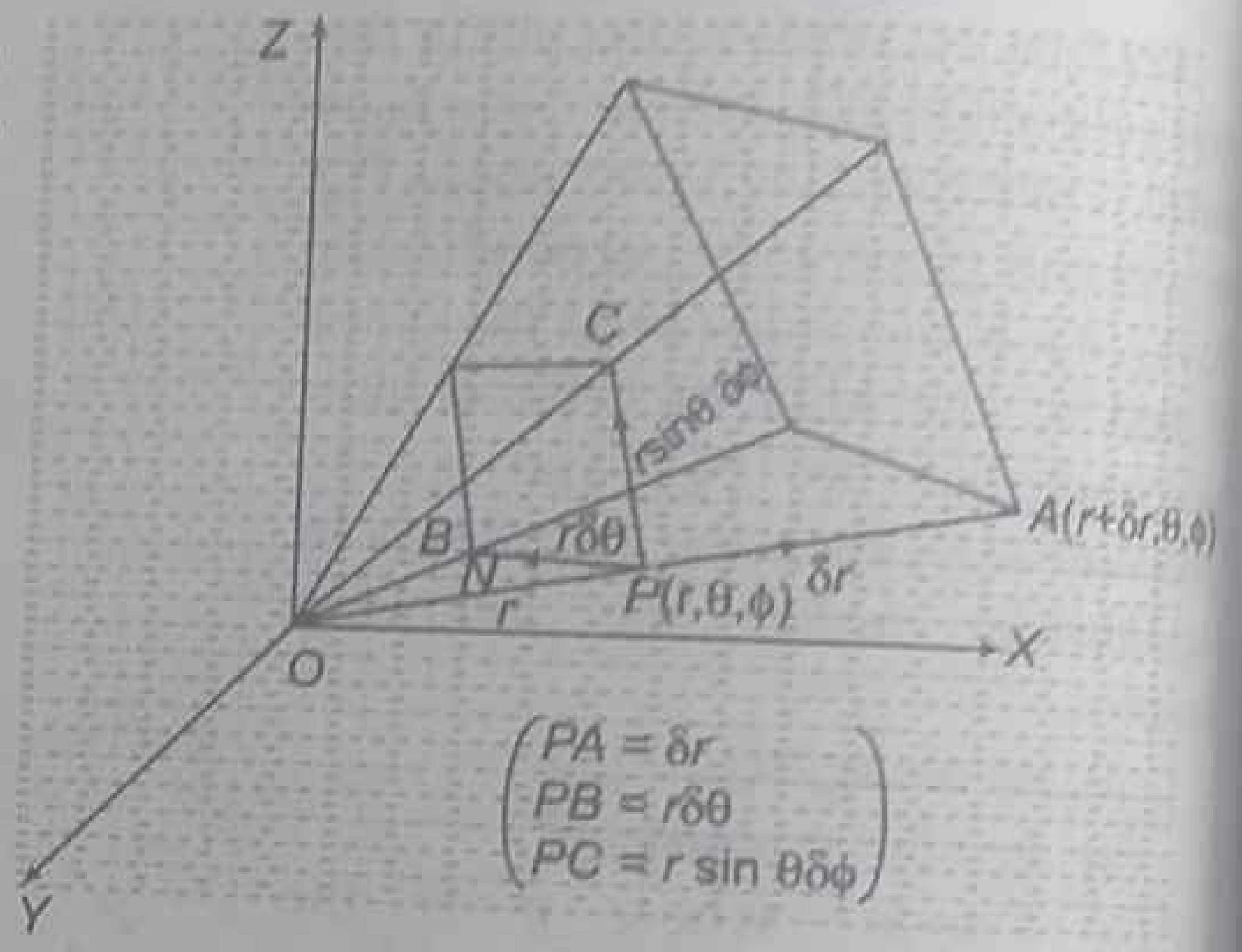

Fig. 163

$$\therefore \quad \frac{\partial p}{r\,\partial\theta} = \rho N \quad \text{or} \quad \frac{\partial p}{\partial\theta} = \rho r N. \qquad \ldots(2)$$

Again the force T is along ϕ increasing, *i.e.*, along PC, the element δs in this direction being $r \sin\theta\ \delta\phi$

$$\therefore \quad \frac{\partial p}{r \sin\theta\ \partial\phi} = \rho T$$

or

$$\frac{\partial p}{\partial\phi} = r \sin\theta\, \rho T. \qquad \ldots(3)$$

Now as p is the function of r, θ, ϕ, we have from (1), (2) & (3)

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial\theta} d\theta + \frac{\partial p}{\partial\phi} d\phi = \rho\,[R\,dr + rN\,d\theta + r\ \sin\theta.\,T d\phi].$$

i.e.,

$$dp = P(Rh + rNd\theta + r\ \sin\theta\ Td\theta) \qquad \ldots(4)$$

This is the required pressure equation in polar coordinates.

Necessary and Sufficient Conditions for Equilibrium :

§ 6.6 : *To determine the necessary and sufficient conditions that must be satisfied by a distribution of forces X, Y, Z, so that the fluid may maintain equilibrium.* **[TMBU-2006H, 2008H, 2011(H)]**

Sol. If p is the pressure of the liquid of density ρ at P (x, y, z) then we have

$$\frac{\partial p}{\partial x} = \rho X, \ \frac{\partial p}{\partial y} = \rho Y, \ \frac{\partial p}{\partial z} = \rho Z. \quad \ldots(1)$$

But p is a function of independent variables x, y, z hence, we have

$$\frac{\partial^2 p}{\partial y\,\partial z} = \frac{\partial^2 p}{\partial z\,\partial y}, \ \frac{\partial^2 p}{\partial z\,\partial x} = \frac{\partial^2 p}{\partial x\,\partial z}, \ \frac{\partial^2 p}{\partial x\,\partial y} = \frac{\partial^2 p}{\partial y\,\partial x}. \quad \ldots(2)$$

Now $\dfrac{\partial^2 p}{\partial y\,\partial z} = \dfrac{\partial^2 p}{\partial z\,\partial y}$ gives $\dfrac{\partial}{\partial y}\left(\dfrac{\partial p}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial p}{\partial y}\right)$,

i.e., $$\frac{\partial}{\partial z}(\rho Z) = \frac{\partial}{\partial z}(\rho Y) \text{ from (1),}$$

i.e., $$Z\frac{\partial \rho}{\partial y} + \rho\frac{\partial Z}{\partial y} = Y\frac{\partial \rho}{\partial z} + \rho\frac{\partial Y}{\partial z}$$

or $$\rho\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) = Z\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial z}. \quad \ldots(3)$$

Similarly $\dfrac{\partial^2 p}{\partial z\,\partial x} = \dfrac{\partial^2 p}{\partial x\,\partial z}$ gives

$$\rho\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) = X\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial x}. \quad \ldots(4)$$

and $\dfrac{\partial^2 p}{\partial x\,\partial y} = \dfrac{\partial^2 p}{\partial y\,\partial x}$ gives

$$\rho\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = Y\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial y}. \quad \ldots(5)$$

Multiplying (3), (4), (5) by X, Y, Z and adding, we get

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0, \quad \ldots(6)$$

This is the required necessary condition for equilibrium under given forces.

Condition is sufficient : Let us suppose, (6) be satisfied. But (6) is the condition that $X\,dx + Y\,dy + Z\,dz$ is an exact differential.

$\therefore$ $\rho(X\,dx + Y\,dy + Z\,dz)$ is an exact differential if (6) holds

$$= dp \text{ say}$$

$$= \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz.$$

$\therefore$ $$\frac{\partial p}{\partial x} = \rho X, \ \frac{\partial p}{\partial y} = \rho Y, \ \frac{\partial p}{\partial z} = \rho Z.$$

These are clearly the equations of equilibrium. Thus if (6) holds, the equations of equilibrium follow. This shows that the condition (6) is sufficient.

§ 6.7 : Pressure equation in homogeneous liquids :

We know that the pressure at a point (x, y, z) is given by

$$dp = \rho\,(X\,dx + Y\,dy + Z\,dz), \qquad \ldots(1)$$

where ρ is the density of the liquid at the point (x, y, z) and X, Y, Z are the component forces per unit mass.

Now if a liquid is homogeneous, ρ is constant ; thus from (1) it follows that

$$X\,dx + Y\,dy + Z\,dz = \frac{1}{\rho}\,dp = d\left(\frac{p}{\rho}\right) \qquad \ldots(2)$$

i.e., $X\,dx + Y\,dy + Z\,dz$ is a perfect differential.

But if $X\,dx + Y\,dy + Z\,dz$ is a perfect differential, then the system of forces is said to be conservative. Therefore a homogeneous liquid will be in equilibrium only when the system of forces is conservative. Thus in this case we may write

$$X\,dx + Y\,dy + Z\,dz = -\,dV \text{ (} V \text{ being potential function)},$$

or $$\frac{dp}{\rho} = -\,dV \text{ from (2)}$$

Integrating, $\dfrac{p}{\rho} + V = C$ (const. of integration).

Note : When ρ = constant, *i.e.*, when the liquid is homogeneous, we have from (3), (4) and (5) of § 6·6, the conditions of equilibrium as

$$\frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z},\ \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x},\ \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}.$$

§ 6.8 : Pressure equation in heterogeneous liquids :

In a heterogeneous liquid, density ρ varies, *i.e.*, ρ is a function of independent variables x, y, z. In this case the system of forces X, Y, Z may maintain equilibrium if (see § 6·6)

$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y),\ \frac{\partial}{\partial z}(\rho X) = \frac{\partial}{\partial x}(\rho Z).\ \frac{\partial}{\partial x}(\rho Y) = \frac{\partial}{\partial y}(\rho X).$$

$$\therefore \quad \frac{\partial \rho}{\partial y} Z + \rho \frac{\partial Z}{\partial y} = \frac{\partial \rho}{\partial z} Y + \rho \frac{\partial Y}{\partial z}$$

$$\frac{\partial \rho}{\partial z} X + \rho \frac{\partial X}{\partial z} = \frac{\partial \rho}{\partial y} Z + \rho \frac{\partial Z}{\partial x}$$

$$\frac{\partial \rho}{\partial x} Y + \rho \frac{\partial Y}{\partial z} = \frac{\partial \rho}{\partial y} X + \rho \frac{\partial X}{\partial y}$$

$$\Rightarrow \quad \left.\begin{aligned} \rho\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) &= Y\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial y} \\ \rho\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) &= Z\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial z} \\ \rho\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) &= X\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial x} \end{aligned}\right\} \qquad \ldots(1)$$

The system of equations (1) represents the conditions of equilibrium of a heterogeneous liquid under given forces.

6.9 : Surfaces of Equal Pressure

[TMBU-2008H, 2011(H)]

For a liquid at rest, the pressure at any point is given by

$$dp = \rho\,(X\,dx + Y\,dy + Z\,dz),$$

which on integration gives $p = \phi\,(x, y, z)$.

If p = constant = C (say), then we have a surface

$$\phi\,(x, y, z) = C \qquad \ldots(1)$$

at every point of which the pressure is constant and equal to C.

Such a surface is called surface of equal pressure.

For different values of C we obtain different surfaces; at every point of these surfaces, the pressure is same. Thus as C takes different values, (1) represents surfaces of equal pressure.

The external surface, or free surface, is obtained by making p equal to the pressure external to the liquid. If external pressure is zero, the free surface is given by

$$\phi\,(x, y, z) = 0.$$

6.10 : Lines of Force

A line of force is a curve (i.e., line) such that the tangent at every point of it is in the direction of the resultant force at that point.

From Coordinate Geometry, we know that the direction-cosines of the tangent at a point (x, y, z) are proportional to

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$$

Also the line, along which the resultant of forces X, Y, Z parallel to axes act has direction-cosines proportional to X, Y, Z.

For line of force these directions are the same

$$\therefore \quad \frac{dx/ds}{X} = \frac{dy/ds}{Y} = \frac{dz/ds}{Z},$$

i.e.,

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

Therefore these are the differential equations of the lines of force.

Example. *Show that in a conservative field of forces the surfaces of equi-pressure and equi-potential energy coincide.* **[CU 2000H, 02H, 04H, MU-2000H]**

6.11 : An Important Property

Remark : The surface of equal pressure are intersected orthogonally by the lines of force.

A surface of equal pressure is $p = \phi\,(x, y, z)$ = const.

And at any point (x, y, z) of this surface, direction cosines of the normal are proportional to $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$,

i.e., proportional to $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ as $\frac{\partial \phi}{\partial x} = \frac{\partial p}{\partial x}$ etc., *i.e.,* proportional to $\rho X, \rho Y, \rho Z$ as $\frac{\partial p}{\partial x} = \rho X$ etc.,

i.e., proportional to X, Y, Z,

which are also direction ratios of lines of force. Thus the lines of force are parallel to normals to the surfaces of equal pressure. Therefore the surfaces of equal pressure are cut orthogonally by lines of force.

Curves of equal pressure and density

[TMBU-2013H]

§ 6.12 : *If a fluid is at rest under the forces X, Y, Z per unit mass, then to find the differential equations of the curves of equal pressure and density.*

Let the fluid be heterogeneous (so that density is a function of (x, y, z)) and incompressible. Then the surfaces of equal pressure are given by $p = \text{const.}$ *i.e.,* $dp = 0$,

or
$$X\,dx + Y\,dy + Z\,dz = 0 \qquad \ldots(1)$$

And surface of equal density are given by

$$\rho = \text{const. } i.e., \ d\rho = 0$$

or
$$\frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz = 0. \qquad \ldots(2)$$

Curves of intersection of (1) and (2) are the curves of equal pressure and density. From (1) and (2),

$$\frac{dx}{Z\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial z}} = \frac{dy}{X\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial x}} = \frac{dz}{Y\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial y}} \qquad \ldots(3)$$

But from the conditions of equilibrium, *i.e.,* from (3), (4) and (5) of (§ 6.6), we have

$$Z\frac{\partial \rho}{\partial y} - Y\frac{\partial \rho}{\partial z} = \rho\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right),$$

$$X\frac{\partial \rho}{\partial z} - Z\frac{\partial \rho}{\partial x} = \rho\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right),$$

and
$$Y\frac{\partial \rho}{\partial x} - X\frac{\partial \rho}{\partial y} = \rho\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)$$

With the help of these conditions, (3) becomes

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

which are the differential equations of the curves of equal pressure and density.

Worked out Examples

(*Problems are of P.G. Standard.)

Example 1. *A liquid of given volume V is at rest under the forces $X = -\frac{\mu x}{a^2}, Y = -\frac{\mu y}{b^2}, Z = -\frac{\mu z}{c^2}$; find the pressure at any point of the liquid and the surfaces of equal pressure.*

Sol. We have $X = -\frac{\mu x}{a^2}, Y = -\frac{\mu y}{b^2}, Z = -\frac{\mu z}{c^2}$

∴ Pressure at any point (x, y, z) is given by

$$dp = \rho\,(X\,dx + Y\,dy + Z\,dz)$$

$$= -\rho\mu\left(\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz\right)$$

Integrating,

$$p = C - \frac{1}{2}\rho\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

The surface of zero pressure, *i.e.*, the free surface is given by

$$p = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2C}{\rho\mu} = \alpha \text{ (say)}$$

$$\Rightarrow \quad \frac{x^2}{a^2\alpha} + \frac{y^2}{b^2\alpha} + \frac{z^2}{c^2\alpha} = 1$$

This is an ellipsoid whose volume is given to be V, therefore

$$V = \frac{4}{3}\pi\sqrt{a^2\alpha}\,\sqrt{b^2\alpha}\,\sqrt{c^2\alpha}$$

$$= \frac{4}{3}\pi\, abc\, \alpha^{3/2}$$

$$\Rightarrow \quad V = \frac{4}{3}\pi abc\left(\frac{2C}{\rho\mu}\right)^{3/2}$$

or

$$C = \frac{\mu\rho}{2}\left(\frac{3V}{4\pi abc}\right)^{2/3};$$

$$\therefore \quad p = \frac{\mu\rho}{2}\left(\frac{3V}{4\pi abc}\right)^{2/3} - \frac{1}{2}\rho\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right).$$

This gives pressure at any point (x, y, z).

Surfaces of equal pressure are obtained by putting

$$p = \text{const.} = A \text{ (say)}.$$

∴ Surfaces of equal pressure are

$$A = C - \frac{1}{2}\rho\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2\,(C - A)}{\rho\mu} = k^2 \text{ (say)},$$

which represent similar ellipsoids.

Example 2. *Prove that; if the force per unit of mass at (x, y, z) parallel to the axes are $y(a-z), x(a-z), xy$, the surfaces of equal pressure are hyperbolic paraboloids and the curves of equal pressure and density are rectangular hyperbolas.* [CU-2000H]

Sol. As given, we have

$$X = y(a-z),\ Y = x(a-z),\ Z = xy,$$

so that $\dfrac{\partial X}{\partial y} = a - z,\ \dfrac{\partial X}{\partial z} = -y,\ \dfrac{\partial Y}{\partial x} = (a-z),\ \dfrac{\partial Y}{\partial z} = -x,$

$$\frac{\partial Z}{\partial x} = y \text{ and } \frac{\partial Z}{\partial y} = x.$$

The condition of equilibrium is

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0,$$

i.e., $$y(a-z)(-x-x) + x(a-z)(y+y) + xy[a-z-(a-z)] = 0,$$

which is clearly satisfied.

Now surfaces of equal pressure are given by $p = \text{const.}$

i.e., $$dp = 0 \text{ or } X\,dx + X\,dy + Z\,dz = 0$$

or $$y(a-z)\,dx + x(a-z)\,dy + xy\,dz = 0$$

putting values of X, Y, Z.

Dividing by $xy(a-z)$, the surfaces of equal pressure are given by

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{a-z} = 0.$$

Integrating,

$$\log x + \log y - \log(a-z) = \log C$$

or $$\frac{xy}{a-z} = C.$$

which are clearly hyperbolic paraboloids,

Again the curves of equal pressure and equal density are given by

$$\frac{dx}{\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y}} = \frac{dy}{\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z}} = \frac{dz}{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}$$

or $$\frac{dx}{-x-x} = \frac{dy}{y+y} = \frac{dz}{(a-z)-(a-z)}$$

or $$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{0}.$$

The last fraction gives $dz = 0$ or $z = \text{const.}$

And the first two fractions give

$$\frac{dx}{x} + \frac{dy}{y} = 0 \text{ or } xy = \text{const.}$$

Thus the curves of equal pressure and equal density are given by

$$xy = \text{const.}, z = \text{const.},$$

which are clearly rectangular hyperbolas.

Alister : We can also proceed as follows :

We have
$$dp = \rho\,[X\,dx + Y\,dy + Z\,dz]$$
$$= \rho\,[y\,(a-z)\,dx + x\,(a-z)\,dy + xy\,dz]$$
$$= \rho\,[(a-z)\,(y\,dx + x\,dy) + xy\,dz]$$
$$= \rho\,[(a-z)\,d\,(xy) - xy\,d\,(a-z)]$$
$$= \rho\,(a-z)^2 \left[\frac{(a-z)\,d\,(xy) - xy\,d\,(a-z)}{(a-z)^2}\right]$$

multiplying Num. and Deno. by $(a-z)^2$

$$= \rho\,(a-z)^2\, d\left(\frac{xy}{a-z}\right) \qquad \ldots(1)$$

Since the fluid is in equilibrium under the given forces, the right hand side must be a perfect differential.

From (1) it is clear that (1) will be a perfect differential if

$$\rho\,(a-z)^2 = \text{const.} = c \text{ (say)}, \qquad \ldots(2)$$

and then
$$dp = cd\left(\frac{xy}{a-z}\right)$$

or
$$p = c' + c\,\frac{xy}{a-z}.$$

Now the surfaces of equal pressure are given by $p = \text{const.}$

or
$$c' + c\,\frac{xy}{a-z} = \text{const.}$$

or
$$\frac{xy}{a-z} = \text{const.} \qquad \ldots(3)$$

which are clearly hyperbolic-paraboloids,

Again when $\rho = \text{const.}$, we get from (2)

$$(a-z)^2 = \text{const.}, \textit{i.e.}, z = \text{const.}$$

When $z = \text{const.}$, (3) gives $\quad xy = \text{const.}$

Thus curves of equal pressure and equal density are

$$xy = \text{const.}, z = \text{const.}$$

which are clearly rectangular hyperbolas.