Example 3. *If* $X = y(y+z)$, $Y = z(z+x)$, $Z = y(y-x)$, *surfaces of equal pressure are the hyperbolic paraboloids* $y(x+z) = c(y+z)$ *and the curves of equal pressure and density are given by* $y(x+z) = \text{const.}$, $(y+z) = \text{const.}$

Sol. We know that

$$dp = \rho\,[X\,dx + Y\,dy + Z\,dz].$$

Now for surfaces of equal pressure, $p = \text{const.}$, *i.e.*, $dp = 0$.

Thus surfaces of equal pressure are given by $dp = 0$

$$X\,dx + Y\,dy + Z\,dz = 0$$

or

$$y\,(y+z)\,dx + z\,(z+x)\,dy + y\,(y-x)\,dz = 0$$

or

$$\frac{dx}{z+x} + \frac{z\,dy}{y\,(y+z)} + \frac{(y-x)\,dz}{(y+z)\,(z+x)} = 0$$

or

$$\frac{dx}{z+x} + \frac{z\,dy}{y\,(y+z)} + \frac{(y+z)-(x+z)}{(y+z)\,(z+x)}\,dz = 0,$$

as $y - x = (y+z) - (x+z)$

or

$$\frac{dx}{z+x} + \frac{z\,dy}{y\,(y+z)} + \frac{dz}{z+x} - \frac{dz}{y+z} = 0$$

or

$$\frac{dx+dz}{x+z} + \frac{z\,dy - y\,dz}{y\,(y+z)} = 0$$

or

$$\frac{dx+dz}{x+z} + \frac{dy\,(z+y) - y\,(dy+dz)}{y\,(y+z)} = 0$$

or

$$\frac{dx+dz}{x+z} + \frac{dy}{y} - \frac{dy+dz}{y+z} = 0$$

Integrating, $\log(x+z) + \log y - \log(y+z) = \log c$

or

$$y\,(x+z) = c\,(y+z),$$

which is the equation giving surfaces of equal pressure.

Again curves of equal pressure and density are given by

$$\frac{dx}{\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y}} = \frac{dy}{\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z}} = \frac{dz}{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}$$

or

$$\frac{dx}{(2z+x)-(2y-x)} = \frac{dy}{-y-y} = \frac{dz}{(2y+z)-z}$$

or

$$\frac{dx}{x+z-y} = \frac{dy}{-y} = \frac{dz}{y}. \quad \text{...(1)}$$

From the last two fractions,

$$dy + dz = 0 \quad \text{or} \quad y + z = \text{const.}$$

Also using the multipliers y, $x + z$, y respectively, we have

$$y\,dx + (x + z)\,dy + y\,dz = 0$$

or

$$y\,dx + x\,dy + z\,dy + y\,dz = 0$$

or

$$xy + yz = \text{const.} \qquad \ldots(2)$$

(1) and (2) together represent curves of equal pressure and density

Example 4. *Show that the forces represented by*

$$X = \mu\,(y^2 + yz + z^2),\ Y = \mu\,(z^2 + zx + x^2),\ Z = \mu\,(x^2 + xy + y^2)$$

will keep a mass of liquid at rest, if the density $\propto \dfrac{1}{(\text{dist.})^2}$ *from the plane* $x + y + z = 0$; *and the curves of equal pressure and density will be circles.*

Sol. Let d be the distance of point (x, y, z) from the plane $x + y + z = 0$; then $d = \dfrac{x + y + z}{\sqrt{3}}$.

If ρ be the density and $\rho \propto \dfrac{1}{(\text{dist.})^2}$, then

$$\rho \propto \frac{1}{d^2} \quad \text{or} \quad \rho = \frac{\lambda}{(x + y + z)^2}, \qquad \ldots(1)$$

where λ is a const, the liquid is heterogeneous.

Now the given forces will keep the heterogeneous fluid in equilibrium, if

$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y),\ \frac{\partial}{\partial x}(\rho Z) = \frac{\partial}{\partial z}(\rho X),\ \frac{\partial}{\partial y}(\rho X), = \frac{\partial}{\partial x}(\rho Y).$$

We have to show that these conditions are satisfied by the given values of X, Y, Z when ρ is given by (1).

Now

$$\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial y}\left[\frac{\lambda}{(x + y + z)^2} \times \mu\,(x^2 + xy + y^2)\right]$$

$$= \lambda\mu\,\frac{(x + 2y)(x + y + z)^2 - 2\,(x + y + z)(x^2 + xy + y^2)}{(x + y + z)^4}$$

$$= \lambda\mu\,\frac{[(x + 2y)(x + y + z) - 2\,(x^2 + xy + y^2)]}{(x + y + z)^3}$$

$$= \frac{\lambda\mu}{(x + y + z)^3}[-x^2 + 2yz + xz + xy]. \qquad \ldots(2)$$

and

$$\frac{\partial}{\partial z}(\rho Y) = \frac{\partial}{\partial z}\left[\frac{\lambda}{(x + y + z)^2}\,\mu\,(z^2 + zx + x^2)\right]$$

$$= \lambda\mu\,\frac{(2z + x)(x + y + z)^2 - 2\,(x + y + z)(z^2 + zx + x^2)}{(x + y + z)^4}$$

$$= \lambda\mu\,\frac{(2z + x)(x + y + z) - 2\,(z^2 + zx + x^2)}{(x + y + z)^3}$$

$$= \frac{\lambda\mu}{(x + y + z)^3}[-x^2 + zx + 2yz + xy]. \qquad \ldots(3)$$

Thus from (2) and (3), we get $\frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y)$.

Similarly it can be shown that

$$\frac{\partial}{\partial x}(\rho Z) = \frac{\partial}{\partial z}(\rho X) \text{ and } \frac{\partial}{\partial y}(\rho X) = \frac{\partial}{\partial x}(\rho Y).$$

Thus the given forces will keep the mass of liquid at rest if ρ is given by (1).

Again the curves of equal pressure and density are

$$\frac{dx}{\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}} = \frac{dy}{\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}} = \frac{dz}{\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}}$$

or $$\frac{dx}{(x+2z)-(x+2y)} = \frac{dy}{(2x+y)-(y+2z)} = \frac{dz}{(2y+z)-(z+2x)}$$

or $$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

$$= \frac{dx+dy+dz}{0} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

Now $dx + dy + dr = 0$ gives $x + y + z = \text{const.}$ ...(4)

and $x\,dx + y\,dy + z\,dz = 0$ gives $x^2 + y^2 + z^2 = \text{const.}$ ...(5)

(4) and (5) together represent the curves of equal pressure and density. These curves being the curves of intersection of planes and spheres represent circles.

Aliter. Alternatively we can proceed as follows :

$$dp = \rho[X\,dx + Y\,dy + Z\,dz]$$

$$= \rho\mu\,[(y^2 + yz + z^2)\,dx + (z^2 + zx + x^2)\,dy + (x^2 + xy + y^2)\,dz]$$

$$= \rho\mu\,[\Sigma\,\{(x+y+z)(y+z) - x(y+z) - zy\}\,dx]$$

$$= \rho\mu\,[(x+y+z)\{(y+z)\,dx + (x+z)\,dy + (x+y)\,dz\} - (yz+zx+xy)(dx+dy+dz)]$$

$$= \rho\mu\,[(x+y+z)\,d\,(xy+yz+zx) - (yz+zx+xy)\,d\,(x+y+z)]$$

$$= \rho\mu\,(x+y+z)^2\,\frac{(x+y+z)\,d\,(xy+yz+zx) - (yz+zx+xy)\,d\,(x+y+z)}{(x+y+z)^2}$$

$$= \rho\mu\,(x+y+z)^2\,d\left(\frac{yz+zx+xy}{x+y+z}\right)$$

The right hand side will be a perfect differential if

$$\rho\mu\,(x+y+z)^2 = k, \text{ a constant}$$

or $$\rho = \frac{k}{\mu\,(x+y+z)^2} = \frac{k/3}{\mu\,[(x+y+z)/\sqrt{3}]^2}$$

or $\rho = \frac{k}{3\mu d} \propto \frac{1}{d^2}$ where $d = \frac{x+y+z}{\sqrt{3}}$ = distance of (x, y, z) from the plane $x + y + z = 0$.

Thus the equilibrium is possible when $p \propto \frac{1}{(\text{dist.})^2}$ from $x + y + z = 0$.

Again ρ = const. gives $x + y + z$ = const. ...(6)

and p = const. gives $dp = 0$, *i.e.*, $d\left(\dfrac{yz + zx + xy}{x + y + z}\right) = 0$

or $\dfrac{yz + zx + xy}{x + y + z}$ = const. or $yz + zx + xy$ = const. by (6).

Thus curves of equal pressure and density are

$$x + y + z = 0,\ yz + zx + xy = \text{const.}$$

or

$$x + y + z = 0,\ x^2 + y^2 + z^2 = \text{const.}$$

which are clearly circles.

Example 5. *A fluid rests in equilibrium in a field of force, $X = y^2 + z^2 - xy - xz$, $Y = z^2 + x^2 - zy - xy$, $Z = x^2 + y^2 - xz - yz$. Show that the curves of equal pressure and density are a set of circles.*

Sol. The differential equations of curves of equal pressure and density are

$$\frac{dx}{\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y}} = \frac{dy}{\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z}} = \frac{dz}{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}$$

or

$$\frac{dx}{(2z - y) - (2y - z)} = \frac{dy}{(2x - z) - (2z - x)} = \frac{dz}{(2y - x) - (2x - y)}$$

or

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

$$= \frac{dx + dy + dz}{0} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

Now $dx + dy + dz = 0$ gives $x + y + z$ = cosnt.

and $x\,dx + y\,dy + z\,dz = 0$ gives $x^2 + y^2 + z^2$ = const.

These equations together represent a set of circles.

Example 6. *If the components parallel to the axes of the forces acting on the element of fluid at (x, y, z) be proportional to*

$$y^2 + 2\lambda yz + z^2,\ z^2 + 2\mu zx + x^2,\ x^2 + 2\nu xy + y^2,$$

show that if equilibrium be possible, we must have

$$2\lambda = 2\mu = 2\upsilon = 1.$$

[TMBU-2006H]

Sol. Here let $X = k\,(y^2 + 2\lambda yz + z^2)$,

$$Y = k\,(z^2 + 2\mu zx + x^2),\ Z = k\,(x^2 + 2\nu xy + y^2).$$

The fluid will be at rest if

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0 \quad \text{...(1)}$$

or if

$$k^2\,\Sigma\,\{(y^2 + 2\lambda yz + z^2)\,(2z + 2\mu x - 2\upsilon x - 2y)\} = 0$$

or

$$\Sigma\,\{(y^2 + 2\lambda yz + z^2)\}\,\{z - y + (\mu - \upsilon)\,x\} = 0.$$

When $2\lambda = 2\mu = 2\upsilon = 1$, it becomes

$$\Sigma \{(y^2 + yz + z^2)(z - y)\} = 0$$

or

$$\Sigma (y^3 - z^3) = 0$$

or

$$(y^3 - z^3) + (z^3 - z^3) + (x^3 - y^3) = 0$$

which is clearly satisfied.

Thus the condition of equilibrium is satisfied when

$$2\lambda = 2\mu = 2\nu = 1.$$

This proves the result.

Example 7. *A mass of fluid is at rest under the forces*

$$X = (y + z)^2 - x^2, \; Y = (z + x)^2 - y^2, \; Z = (x + y)^2 - z^2;$$

find the density and prove that the surfaces of equal pressure are lypsboloids of revolution.

Sol. Here $X = (y + z)^2 - x^2, \; Y = (z + x)^2 - y^2, \; Z = (x + y)^2 - z^2$

$$\frac{\partial X}{\partial y} = 2(y + z) = \frac{\partial X}{\partial z} \quad \text{and} \quad \frac{\partial Z}{\partial x} = 2(x + y) = \frac{\partial Z}{\partial y}$$

$$\frac{\partial Y}{\partial z} = 2(z + x) = \frac{\partial Y}{\partial x}$$

The condition of equilibrium is

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial z} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)$$

$$= 2[(y + z)^2 - x^2](x - y) + [(z + x)^2 - y^2](x - z) + 2[(x + y)^2 - z^2](y - z)$$

$= 0$ is satisfied.

Again pressure p at a point (x, y, z) is given by

$$dp = \rho[X\,dx + Y\,dy + Z\,dz]$$

$$= \rho[\{(y + z)^2 - x^2\}dx + \{(z + x)^2 - y^2\}dy + \{(x + y)^2 - z^2\}dz]$$

$$= \rho(x + y + z)[(y + z)\,dx + (z + x)\,dy + (x + y)\,dz - x\,dx - y\,dy - z\,dz]$$

$$dp = \frac{\rho}{2}(x + y + z)\,d[2yz + 2zx + 2xy - x^2 - y^2 - z^2] \quad \ldots(1)$$

The right hand side of (1) is a perfect differential, if

$$\rho(x + y + z) = c = a \text{ constant}$$

$$\rho = \frac{c}{x + y + z} \quad i.e., \; \rho \propto \frac{1}{x + y + z}$$

Again putting $p = 0$ then from (1) we get $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = \text{constant} \quad \ldots(2)$

Now to show that (2) represents hyperboloids of revolution, the discriminating cubic is

$$\lambda^3 - (a + b + c)\lambda^2 + (bc + ca + ab - f^2 - g^2 - h^2)x + (af^2 + bg^2 + ch^2 - 2fgh - abc) = 0$$

where $a = b = c = 1, \; f = g = h = -1;$

$\therefore$ The discriminating cubic becomes $\lambda^3 - 3\lambda^2 + 4 = 0$ *i.e.*, $\lambda = 2, 2, -1$. The two roots are equal; hence (2) is a paraboloid of revolution. Further since one value of λ is negative; hence the surface is hyperbolid. Thus the surface of equal pressure are hyperboloids of revolution.

Example 8. *A tank of heavy homogeneous liquid is moved horizontally with uniform acceleration f. Find the pressure at any point.*

Sol. Considering the axis of x along the uniform acceleration f and that of y vertically upwards. The reversed effective force per unit mass along the axis of x that is $X = -f$. The force of gravity per unit mass along the axis of y is given by $Y = -g$.

i.e., With the application of a force with components $X = -f$ and $Y = -g$, the body must be in equilibrium.

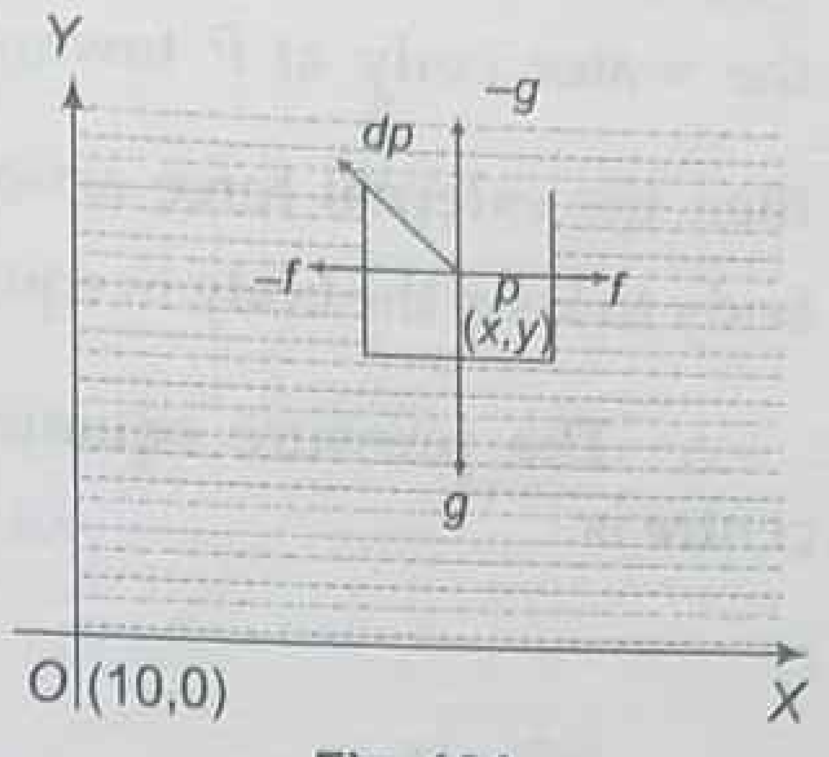

Fig. 164

Hence the pressure equation at any point $P(x, y)$, is

$$dp = \rho\,(X\,dx + Y\,dy) \qquad \ldots(1)$$

$$= \rho\,(-f\,dx - g\,dy)$$

$$dp = -\rho\,(f\,dx + g\,dy)$$

Integrating, we get

$$p = C - \rho\,(fx + gy) \qquad \ldots(2)$$

where C is constant of integration.

On the free surface $p = \Pi$, then the equation of the free surface

$$C - \rho\,(fx + gy) = \Pi \qquad \ldots(3)$$

$$\Rightarrow C - \Pi - \rho fx - \rho gy = 0 \qquad \ldots(4)$$

If z be the distance of any point $P(x, y)$ below the free surface, thus

$$z = \frac{|C - \rho fx - \rho gy - \Pi|}{\sqrt{(\rho f)^2 + (\rho g)^2}}$$

$$z = \frac{C - \rho\,(fx + gx) - \Pi}{\rho\sqrt{f^2 + g^2}}$$

$$\Rightarrow \quad \rho z\sqrt{f^2 + g^2} = C - \rho\,(fx + gy) - \Pi$$

$$\Rightarrow \quad \rho z\sqrt{f^2 + g^2} = p - \pi \text{ using (2)}$$

$$\Rightarrow \quad p = \pi + \rho z\sqrt{f^2 + g^2} \qquad \ldots(5)$$

This is the required pressure of the liquid at a point $P(x, y)$.

Example 9. *Show that the pressure at a small depth z below the surface of a sphere of water attracted to the centre of the sphere with a force an acceleration $\frac{\mu}{r^2}$ at a distance r is approximately $\Pi + \rho g\left(z + \frac{z^2}{a}\right)$, where a is the radius of the sphere and g the attraction of a unit mass at the surface of the sphere.*

Sol. Considering unit mass of water at P at a distance r from the centre of the sphere. The force per unit mass of the water body at P towards $PC = \frac{\mu}{r^2}$, P where $PC = r$, then the external force acting on the unit mass of water body to keep the body in equilibium is $F = -\frac{\mu}{r^2}$

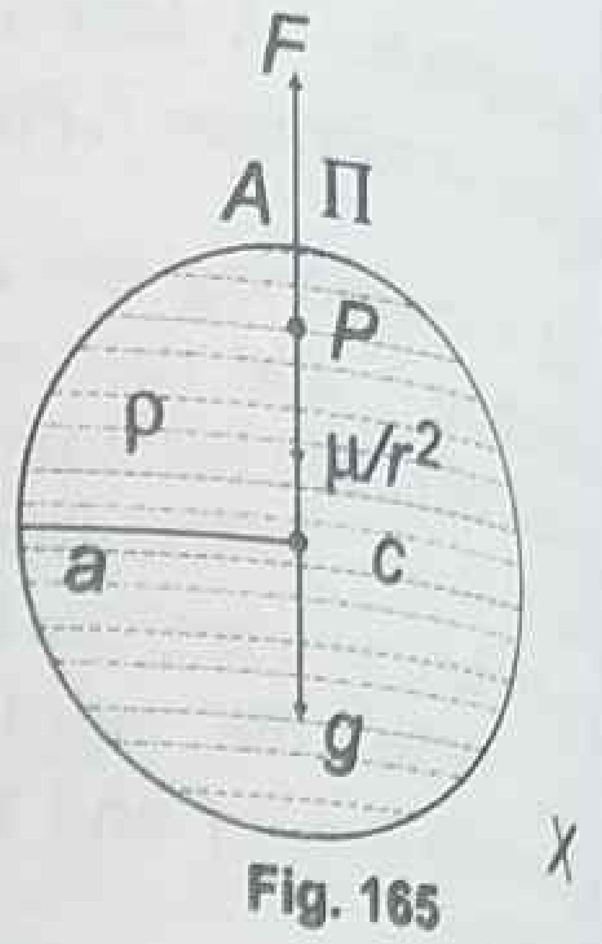

Fig. 165

∴ The pressure equation at a distance r from the centre is

$$\frac{dp}{dr} = \rho F$$

$$dp = \rho F dr = -\frac{\rho\mu}{r^2} dr$$

Integrating, we get

$$p = C + \frac{\rho\mu}{r}$$

At the free surface, $p = \Pi$, when $r = a$, then from (1),

$$\Pi = \frac{\rho\mu}{a} + C$$

Subtracting (2) from (1) we get

$$p - \Pi = \rho\mu\left(\frac{1}{r} - \frac{1}{a}\right)$$

$$\Rightarrow \quad p = \Pi + \rho\mu\left(\frac{1}{r} - \frac{1}{a}\right)$$

At a very small depth z below the surface of sphere, the pressure is given by

$$p = \Pi + \rho\mu\left(\frac{1}{a-z} - \frac{1}{a}\right)$$

$$= \Pi + \frac{\rho\mu z}{a(a-z)} = \Pi + \frac{\rho\mu z}{a^2(1 - z/a)}$$

$$= \Pi + \frac{\rho\mu z}{a^2}\left[1 - \frac{z}{a}\right]^{-1}$$

$$= \Pi + \frac{\rho\mu z}{a^2}\left[1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \ldots\right]$$

$$p = \Pi + \frac{\rho\mu z}{a^2}\left(1 + \frac{2}{a}\right) \quad \text{approximately}$$

[As $\frac{2}{a}$ is very small

Now acceleration at the free surface $= \frac{\mu}{a^2} = g$

$$\therefore \quad p = \Pi + \rho g z\left(1 + \frac{z}{a}\right) = \Pi + \rho g\left(z + \frac{z^2}{a}\right)$$

Example 10. *A given volume of heavy liquid is at rest under the action of a force to a fixed point varying as the distance from that point. Find the pressure at any point.*

Sol. Let the fixed point be taken as origin and the axis of z vertical. The components of the attractive force μr along the axes are

$$-\mu x, -\mu y, -\mu z.$$

The liquid is heavy, hence the acceleration due to gravity acts vertically downwards; therefore the components of force $-\mu r$ on the point $P(x, y, z)$ are

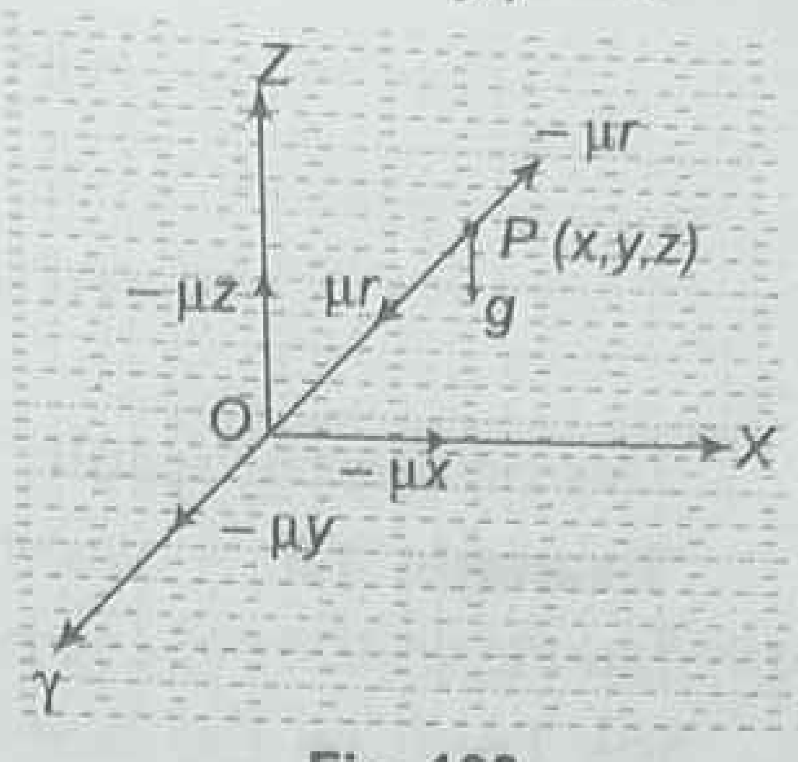

Fig. 166

$$\left.\begin{aligned} X &= -\mu x \\ Y &= -\mu y \\ Z &= -\mu z - g \end{aligned}\right\} \quad \ldots(1)$$

$\therefore$ The pressure at any point is given by

$$dp = \rho\,(X\,dx + \gamma\,dy + Z\,dx)$$

$$= -\rho\mu\left(x\,dx + y\,dy + z\,dz + \frac{g}{\mu}\,dz\right)$$

Integrating, we get

$$p = C - \frac{1}{2}\rho\mu\left(x^2 + y^2 + z^2 + \frac{2gz}{\mu}\right) \quad \ldots(2)$$

This is the required expression for pressure at any point of the liquid.

***Example 11.** *The particles of a sphere of homogeneous fluid of radius a are attracted to the centre with force inversely proportional to the distance from the centre; prove that the pressure at a distance x from the surface (measured along the radius) is*

$$C + D\rho\left(x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \ldots\right),$$

where C is a constant, D is the force per unit mass at the surface of the sphere and ρ is the constant density of the fluid.

Sol. Consider a point P of the fluid at a distance r from the centre of force, then the pressure equation is given by

$$\therefore \quad \frac{dp}{dr} = -\rho F \Rightarrow dp = -\rho\left(\frac{\mu}{r}\right)dr.$$

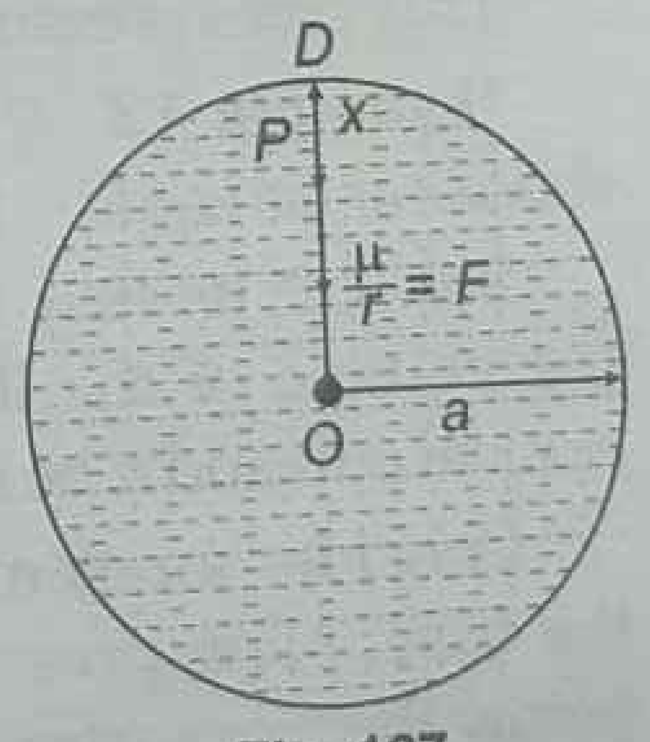

Fig. 167

But force at the surface is D, *i.e.*, $D = \frac{\mu}{a}$ or $\mu = aD$.

$$\therefore \quad dp = -\rho\frac{aD}{r}\,dr.$$

Integrating, $\quad p = -\rho a D \log r + A$ (const.)

Now if x denotes the distance of P from the surface of the sphere, then

$$x = a - r \quad \text{or} \quad r = a - x.$$

$$\therefore \quad p = -\rho a D \log(a - x) + A$$

$$= -\rho a D \log a\left(1 - \frac{x}{a}\right) + A$$

$$= -\rho a D \log a - \rho a D \log\left(1 - \frac{x}{a}\right) + A$$

$$= (A - \rho a D \log a) + \rho a D\left(\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \dots\right)$$

$$= C + \rho D\left(x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \dots\right), \quad \text{where } C = A - \rho a D \log a$$

This proves the result.

***Example 12.** *A closed cylinder, the axis of which is vertical, contains a given mass of air; find the pressure at any point of the fluid.*

Sol. Let h be the height and a the radius of the base of the cylinder.

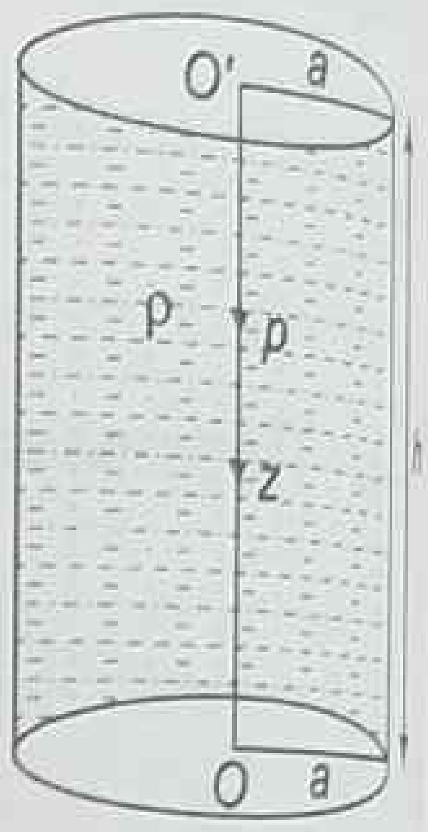

Fig. 168

Consider a point at a height z above the base; there ρ is the density of air, then the pressure at this height is given by

$$dp = \rho g\, dz.$$

Also for air,

$$p = k\rho.$$

$\therefore$

$$\frac{dp}{p} = -\frac{g}{k} dz.$$

Integrating,

$$\log p = \log A - \frac{g}{k} z$$

or

$$p = Ae^{-gz/k}. \quad \dots(1)$$

Now to determine the constant A, we have the air of given mass say M,

$$\therefore \quad M = \int_0^h \pi a^2 \rho\, dz = \int_0^h \pi a^2 \frac{A}{k} e^{-gz/k} dz \quad \text{as } \rho = \frac{p}{k} = \frac{A}{k}e^{-gz/k}$$

$$= \frac{\pi a^2 A}{g} (-e^{-gz/k})_0^h = \frac{\pi a^2 A}{g} (1 - e^{-gh/k})$$

or

$$A = \frac{gM}{\pi a^2} \frac{1}{(1 - e^{-gh/k})}$$

$\therefore$ From (1)

$$p = e^{-gz/k} \frac{gM}{\pi a^2} \frac{1/3}{1 - e^{-gh/k}}.$$

This gives pressure at any point.

***Example 13.** *If a conical cup be filled with liquid, the mean pressure at a point in the volume of the liquid is to the mean pressure at a point in the surface of the cup as 3 : 4.*

Sol. Let us first find mean pressure at a point in the volume of the liquid. Let h be the height of the conical cup and α be its semi-vertical angle.

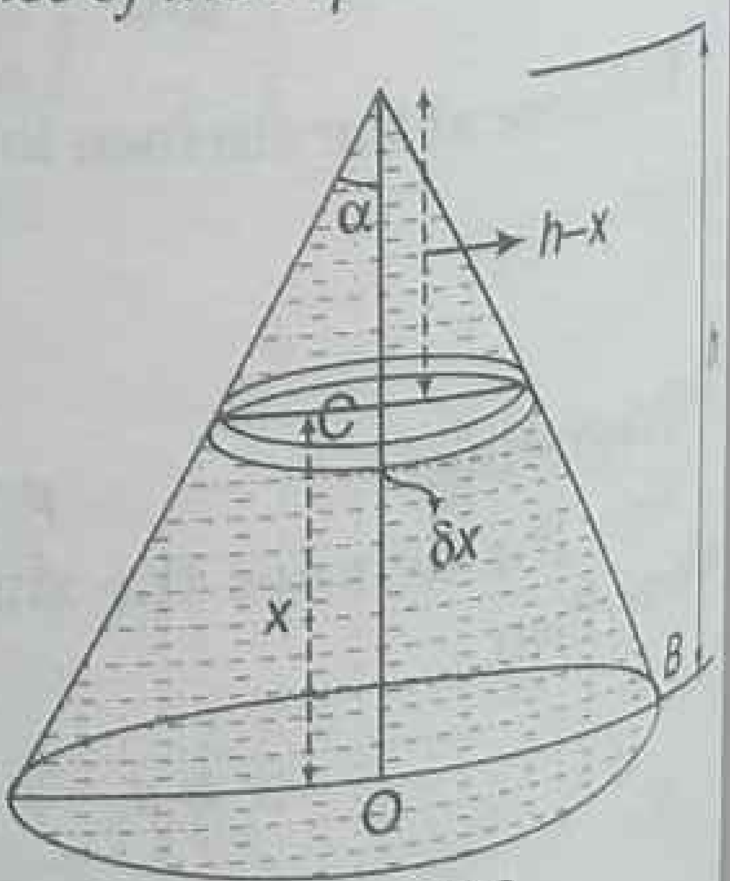

Fig. 169

Consider an elementary disc at a distance x from the base. Now pressure at a point at a depth $(h - x)$ below the free surface

$$= \rho g\,(h - x),$$

dv = volume of the disc

$$= (\pi x^2 \tan^2 \alpha\, \delta x).$$

$\therefore$ $p\,dv$ = pressure on the elementary disc

$$= \rho g\,(h - x)\,(\pi x^2 \tan^2 \alpha\, \delta x).$$

Now mean pressure at a point in the volume,

$$P_1 = \frac{\iint p\,dv}{\int dv} = \frac{\int_0^h \rho g\,(h-x)\,\pi x^2 \tan^2\alpha\,dx}{\int_0^h \pi x^2 \tan^2\alpha\,dx}$$

$$= \rho g \frac{\int_0^h (x^2 h - x^3)\,dx}{\int_0^h x^2\,dx} = \rho g \frac{\frac{1}{3}h^4 - \frac{1}{4}h^4}{\frac{1}{3}h^3} = \frac{1}{4}\rho g h. \quad \ldots(1)$$

Again to find mean pressure at a point in the surface,

$$ds = \text{elementary surface of the disc}$$

$$= 2\pi x \tan\alpha \cdot \delta x \sec\alpha,$$

$p\,ds$ = pressure on the elementary surface of the disc

$$= \rho g\,(h-x)\,2\pi x \tan\alpha \cdot \delta x \sec\alpha.$$

∴ Mean pressure P_2 at a point in the surface is

$$P_2 = \frac{\int p\,ds}{\int ds} = \frac{\int_0^h \rho g\,(h-x)\,2\pi x \tan\alpha\,dx \sec\alpha}{\int 2\pi x \tan\alpha\,dx \sec\alpha}$$

$$= \rho g \frac{\int_0^h (xh - x^2)\,dx}{\int_0^h x\,dx} = \rho g \frac{\frac{1}{2}h^3 - \frac{1}{3}h^3}{\frac{1}{2}h^2} = \frac{1}{3}\rho g h. \quad \ldots(2)$$

$$\therefore \quad \frac{P_1}{P_2} = \frac{\text{Mean pressure on the volume}}{\text{Mean pressure on the surface}} = \frac{1/4\,\rho g h}{1/3\,\rho g h} = \frac{3}{4}.$$

This proves the result.

*Example 14. *A given volume of liquid is at rest on a fixed plane under the action of a force, to a fixed point in the plane, varying as the distance. Find the pressure at any point of the liquid and the whole pressure on the fixed plane.*

Sol. Let the fixed point be taken as origin. Consider a point P at a distance r from the origin. Force on it $= \mu r$ towards the centre; hence pressure at this point is given by

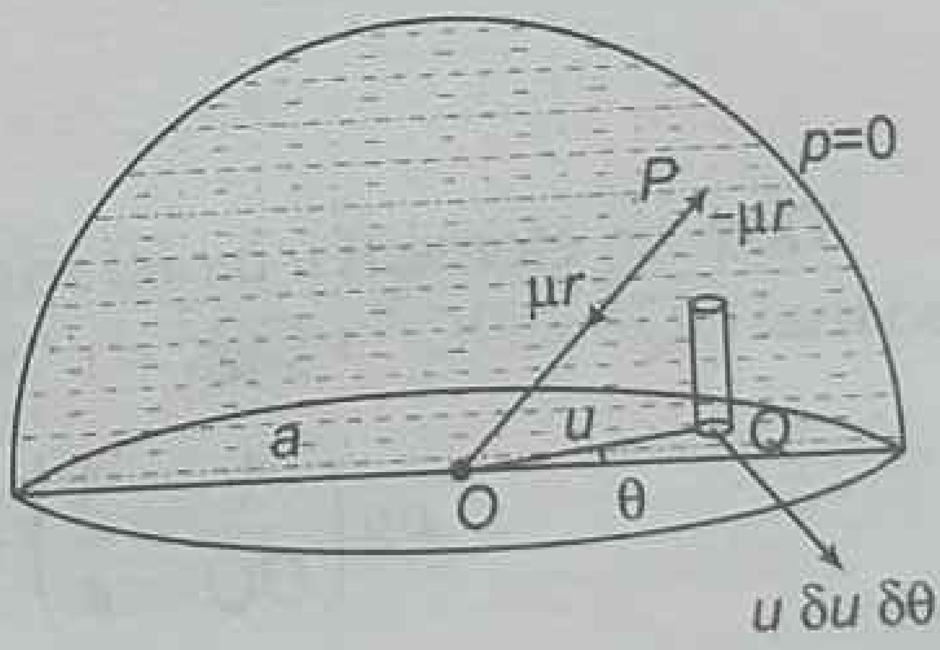

Fig. 170

$$dp = -\rho\mu\,r\,dr.$$

Integrating, $p = c - \frac{1}{2}\rho\mu r^2$.

And if $2/3\,\pi a^3$ be the given volume, the free surface is a hemisphere of radius a. Thus we have

$$p = 0 \text{ when } r = a$$

$$\therefore \quad c = \frac{1}{2}\rho\mu a^2$$

$$\therefore \quad p = \frac{1}{2}\rho\mu\,(a^2 - r^2).$$

This gives pressure at any point P of the liquid.

To find whole pressure on the fixed plane. To find total pressure on the fixed plane (which is here a circle of radius a) consider a point Q on the plane at a distance u from O enclosed by an element of area $u\,\delta u\,\delta\theta$.

Putting $r = u$ in (1), pressure at $Q = \frac{1}{2}\rho\mu\ (a^2 - \mu^2)$.

$\therefore$ Pressure on the element $u\ \delta u\ \delta\theta = \frac{1}{2}\rho\mu\ (a^2 - \mu^2)\ u\ \delta u\ \delta\theta$.

So total pressure $= \int_0^{2\pi} \int_0^a \frac{1}{2}\rho\mu\ (a^2 - u^2)\ u\ du\ d\theta$

$$= \pi\rho\mu \left[\frac{1}{2}a^2u^2 - \frac{1}{4}u^4\right]_0^a = \pi\rho\mu\left[\frac{1}{2}a^4 - \frac{1}{4}a^4\right] = \frac{1}{4}\pi\,\rho\mu a^4.$$

***Example 15.** *A mass of liquid rests upon a plane subject to a central attractive force μ/r^2, situated at a distance c from the plane on the side opposite to that on which is the fluid; show that the pressure on the plane* $= \dfrac{\pi\rho\mu\ (a-c)^2}{a}$.

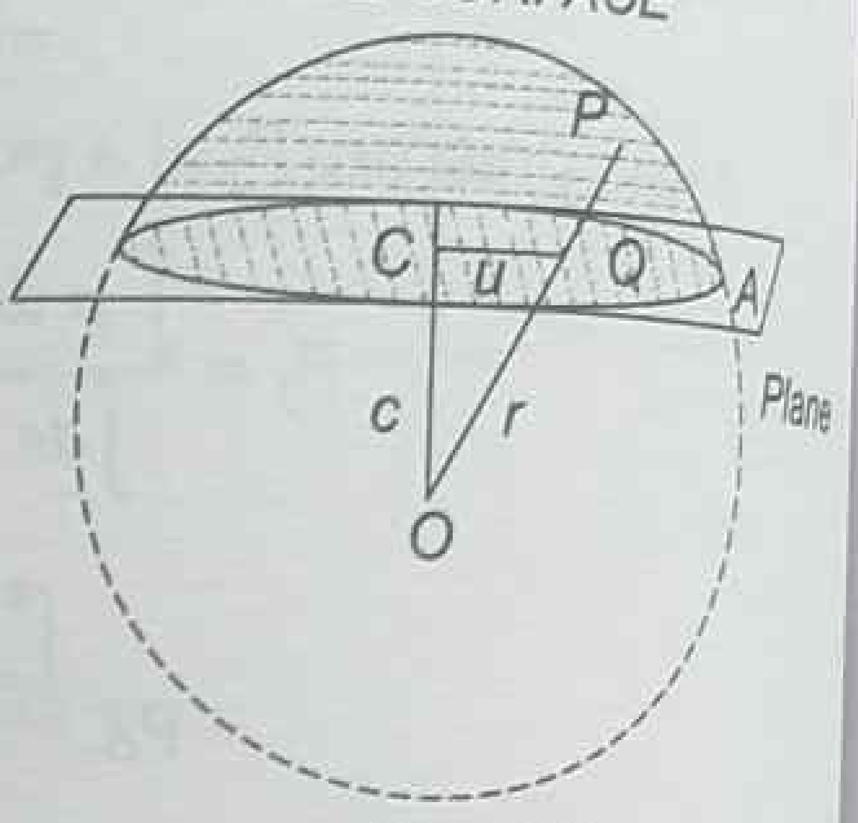

Fig. 171

Sol. The liquid rests on the plane in the form of a cap which is part of a sphere of radius a. We have to determine pressure on the plane (the circular base of the cap). Let us first consider a point P in the liquid at a distance r from the centre O of the sphere.

The only force on P is $\frac{\mu}{r^2}$ towards O. Therefore the pressure at P is given by $dp = -\rho\frac{\mu}{r^2}\,dr$.

Integrating $p = c + \frac{\rho\mu}{r}$

But when $r = a$, $p = 0$ at the free surface ; $\therefore c = -\frac{\rho\mu}{a}$

Thus
$$p = \rho\mu\left(\frac{1}{r} - \frac{1}{a}\right) = \rho\mu\left(\frac{1}{OP} - \frac{1}{a}\right)$$

Now consider a point Q on the plane base of the cap ; then pressure at Q

$$= \rho\mu\left(\frac{1}{OQ} - \frac{1}{a}\right).$$

Take an element $u\ \delta u\ \delta\theta$ surrounding the point Q such that $CQ = u$ and $\angle COQ = \theta$, so that $OQ^2 = OC^2 + CQ^2 = c^2 + u^2$; then pressure on the small element $u\ \delta u\ \delta\theta$ at Q

$$\rho\mu\left(\frac{1}{OQ} - \frac{1}{a}\right) + u\ \delta u\ \delta\theta = \rho\mu\left\{\frac{1}{\sqrt{(c^2+u^2)}} - \frac{1}{a}\right\} u\ \delta u\ \delta\theta.$$

Integrating between the proper limits, so as to include the whole circular base of the cap. total pressure on the plane.

$$= \rho\mu \int_0^{\sqrt{(a^2-c^2)}} \int_0^{2\pi} \left\{\frac{1}{\sqrt{(c^2+u^2)}} - \frac{1}{a}\right\} u\ du\ d\theta,$$

[radius of the base being $\sqrt{a^2 - c^2}$]

$$= \rho\mu\,[\theta]_0^{2\pi}\left[\sqrt{(c^2+u^2)} - \frac{u^2}{2a}\right]_0^{\sqrt{(a^2-c^2)}}$$

$$= 2\pi\rho\mu\left(a - c - \frac{a^2 - c^2}{2a}\right)$$

$$= 2\pi\rho\mu\,(a - c)\left(1 - \frac{a + c}{2a}\right) = \frac{\pi\rho\mu}{a}(a - c)^2.$$