

*Example 20. The density of a liquid contained in a cylindrical vessel, varies as the depth; it is transferred to another vessel, in which the density varies as the square of the depth; find the shape of the new vessel.

Sol. Let A be the area of the cross-section of the cylindrical vessel and h the depth to which liquid is originally filled in it. If V be the volume and M the mass of the liquid, then

$$V = Ah \text{ and } M = \int_0^h A dz \cdot \lambda z = A\lambda \frac{1}{2}h^2. \quad \dots(1)$$

Again suppose the liquid is now transferred to a vessel whose shape * is given by $u^2 = f'(z)$, where $u^2 = x^2 + y^2$.

If the liquid stands in this vessel to a height h' , then since the volume of the liquid remains unchanged.

$$V = Ah = \int_0^{h'} \pi u^2 dz = \int_0^{h'} \pi f'(z) dz = \pi [f(z)]_0^{h'}. \quad \dots(2)$$

or

$$Ah = \pi [f(h') - f(0)]. \quad \dots(2)$$

Differentiating it w.r.t. h' ,

$$A \frac{dh}{dh'} = \pi [f'(h')]. \quad \dots(3)$$

Again the mass M also remains unchanged.

$$M = A\lambda \frac{1}{2}h^2 = \int_0^{h'} \pi u^2 \mu z^2 dz = \pi \mu \int_0^{h'} f'(z) z^2 dz.$$

$$\left[* \int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \right]$$

Differentiating both the sides w.r.t. h' , we have

$$A\lambda h \frac{dh}{dh'} = \mu [f'(h')h'^2]$$

$$\lambda h \cdot \pi f'(h') = \mu \pi f'(h')h'^2$$

$$\lambda h = \mu h'^2 \quad \text{or} \quad \lambda h[f(h') - f'(0)] = A\mu h'^2$$

$$\lambda \pi f'(h') = 2A\mu h'$$

$$f'(h') = \frac{2A\mu}{\lambda\pi} h'$$

$$f'(h') = ch', \text{ where } c = \frac{2A\mu}{\lambda\pi} = \text{const.}$$

$$f'(z) = cz, \text{ putting } z \text{ for } h'.$$

Therefore the shape of the vessel is $u^2 = f'(z)$, i.e., $x^2 + y^2 = cz$, which is a paraboloid of revolution.

Example 21. A compressible fluid is at rest under gravity. Defining compressibility k by the relation $\frac{\rho - \rho_0}{\rho_0} = k(p - p_0)$ where ρ and p are the density and pressure respectively

and p_0 referred to the free surface, and assuming k to be a constant, show that at depth z below the free surface

$$\frac{dp}{dz} = g\rho_0[l + k(p - p_0)]$$

and that $\rho = \rho_0 e^{k\rho_0 z}$

Sol. We have $\frac{\rho - \rho_0}{\rho_0} = k(p - p_0)$,

$$\rho = \rho_0[1 + k(p - p_0)]. \quad \dots(1)$$

This fluid is at rest under gravity; so if axis of z is taken as vertically downwards, we have $X = 0, Y = 0, Z = g$.

∴ the pressure is given by

$$\begin{aligned} dp &= \rho(X dx + Y dy + Z dz) \\ &= \rho g dz = \rho_0[1 + k(p - p_0)] dz. \end{aligned} \quad \dots(2)$$

This gives

$$\frac{dp}{dz} = \rho_0[1 + k(p - p_0)]. \quad \dots(3)$$

Again differentiating (1), $d\rho = \rho_0 k dp$.

$$\therefore (3) \text{ gives } \frac{d\rho}{\rho_0 k} = \rho g dz \quad \text{or} \quad \frac{d\rho}{\rho} = k \rho_0 g dz.$$

Integrating,

$$\log \rho = k \rho_0 g z + \log C$$

$$\rho = C e^{k \rho_0 g z}$$

But when $\rho = \rho_0, z = 0 \therefore A = \rho_0$; thus $\rho = \rho_0 e^{k \rho_0 g z}$.

$$c = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

Pressure of Elastic Fluid

problem: To determine the conditions of equilibrium and pressure at any point.

§ 6.13: To will have to consider two cases :

Sol. Here we temperature remains constant,

when temperature varies.

1. when temperature is constant.

2. when temperature is constant.

When temperature is constant.

We have $p = k\rho$ by Boyle's law.

$$dp = \rho(X dx + Y dy + Z dz)$$

$$= \frac{p}{k} (X dx + Y dy + Z dz),$$

$$\frac{dp}{p} = \frac{1}{k} (X dx + Y dy + Z dz). \quad \dots(1)$$

This gives

Now if the forces are conservative, then

$$X dx + Y dy + Z dz = \text{complete differential} = -dV$$

(where V is the potential function).

$$\therefore (1) \text{ becomes } \frac{dp}{p} = -\frac{dV}{k}$$

Integrating,

$$\log p = -\frac{1}{k}V + \log C \quad \text{or} \quad \frac{p}{C} = e^{-v/k}$$

$$p = C e^{-v/k} \text{ and } \rho = \frac{p}{k} = \frac{C}{k} = e^{-v/k}$$

This gives pressure and density at any point.

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When the temperature varies. We have $p = k\rho T$

$$dp = \rho (X dx + Y dy + Z dz);$$

and

$$dp = \frac{p}{kT} (X dx + Y dy + Z dz)$$

therefore

$$\frac{dp}{p} = \frac{X dx + Y dy + Z dz}{kT}$$

or

This gives pressure at any point.

This gives pressure at any point as given in § 2.5 p. 9.

The condition of equilibrium is same as given in § 2.5 p. 9.
Example 29. A mass M of gas uniform temperature is diffused through all space, and at each point (x, y, z) the components of force per unit mass are $-Ax, -By, -Cz$. The pressure and density at the origin are ρ_0 and p_0 respectively. Prove that

$$ABC\rho_0 M^2 = 8\pi^3 p_0^3.$$

Sol. The gas is an elastic fluid, so we have

$$p = kp \quad \text{or} \quad dp = k d\rho$$

Under the given forces,

$$dp = -\rho [Ax dx + By dy + Cz dz]$$

or

$$kdp = -\rho [Ax dx + By dy + Cz dz].$$

or

$$\frac{dp}{\rho} = -\frac{1}{k} [Ax dx + By dy + Cz dz].$$

Integrating,

$$\log \rho = \log C - \frac{1}{2k} (Ax^2 + By^2 + Cz^2)$$

or

$$\rho = C e^{-\frac{Ax^2 + By^2 + Cz^2}{2k}}.$$

But at origin, i.e., when $x = y = z = 0$, $\rho = \rho_0$; $\therefore C = \rho_0$. Thus

$$\rho = \rho_0 e^{-\frac{Ax^2 + By^2 + Cz^2}{2k}}.$$

Now the mass, M , of the fluid is given by

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho dx dy dz \\ = 8 \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \rho_0 e^{-\frac{Ax^2 + By^2 + Cz^2}{2k}} dx dy dz$$

$$= 8\rho_0 \int_0^{\infty} e^{-\frac{Ax^2}{2k}} dx \int_0^{\infty} e^{-\frac{By^2}{2k}} dy \int_0^{\infty} e^{-\frac{Cz^2}{2k}} dz \\ \text{Putting value of } \rho \text{ from}$$

$$= 8\rho_0 \sqrt{\frac{\pi}{2k}} \cdot \sqrt{\frac{\pi}{2k}} \cdot \sqrt{\frac{\pi}{2k}} \\ = 8\rho_0 \frac{\sqrt{\pi}}{2\sqrt{\left(\frac{A}{2k}\right)}} \cdot \frac{\sqrt{\pi}}{2\sqrt{\left(\frac{B}{2k}\right)}} \cdot \frac{\sqrt{\pi}}{2\sqrt{\left(\frac{C}{2k}\right)}} \quad [\text{following the result of pag}^e] \\ \text{or} \quad M^2 = \frac{\rho_0^2 \pi^3 8k^3}{ABC}$$

$$M^2 ABC = 8\rho_0^2 \pi^3 \left(\frac{p_0^3}{\rho_0^3} \right) \quad \left(\text{as } k = \frac{p_0}{\rho_0} \right)$$

$$ABC \rho_0 M^2 = 8\pi^3 p_0^3.$$

Example 30. The density of a gravitating liquid sphere of radius a at any point increases uniformly as the point approaches the centre. The surface density is ρ_0 and the mean density is ρ . Prove that the pressure at the centre is

$$\frac{2}{9}\pi a^2 \{10\rho(\rho - \rho_0 + 3\rho_0^2)\}$$

Sol. At a point P distant r from the centre, let σ be the density of the liquid, then since the density increases uniformly as r decreases (i.e., as a point approaches centre), we have

$$\frac{d\sigma}{dr} = -\lambda, \lambda \text{ being a constant.}$$

$$\sigma = C = \lambda r.$$

Integrating

But when $r = a$, $\sigma = \rho_0$ at the surface of the sphere.

$$C = \rho_0 - \lambda a; \therefore \sigma = \rho_0 + \lambda(a - r). \quad \dots(1)$$

Again Mean density = $\frac{\text{total mass}}{\text{total volume}}$.

$$\begin{aligned} \rho &= \frac{\int_0^a 4\pi r^2 \sigma dr}{\int_0^a 4\pi r^2 dr} = \frac{\int_0^a r^2 \{\rho_0 + \lambda(a - r)\} dr}{\int_0^a r^2 dr} \\ &= \frac{\rho_0 \cdot \frac{1}{3}a^3 + \lambda \left(\frac{1}{2}a^4 - \frac{1}{4}a^4 \right)}{\frac{1}{3}a^3} = \rho_0 + \frac{\lambda a}{4}. \end{aligned} \quad \dots(2)$$

Thus we have

Now attraction of the liquid at the point P is $\frac{M}{r^2}$

where M = mass of the liquid forming a sphere of radius r

$$\begin{aligned} \int_0^r 4\pi r^2 \sigma dr &= \int_0^r 4\pi r^2 [\rho_0 + \lambda(a - r)] dr \\ &= 4\pi \left[\frac{2}{3}\rho_0 r^3 + \lambda \left(\frac{1}{2}ar^3 - \frac{1}{4}r^4 \right) \right] = \frac{4}{3}r^3 \left[\rho_0 + \lambda \left(a - \frac{3}{4}r \right) \right]. \end{aligned}$$

Therefore the pressure at P is given by

$$\begin{aligned} dp &= -[\rho_0 + \lambda(a - r)] \cdot \frac{4}{3}\pi r^3 \left[\rho_0 + \lambda \left(a - \frac{3}{4}r \right) \right] \frac{1}{r^2} dr \\ &= -\frac{4}{3}\pi r \left[(\rho_0 + \lambda a) - \lambda r \right] \left[(\rho_0 + \lambda a) - \frac{3}{4}\lambda r \right] dr \end{aligned}$$

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$$= -\frac{4}{3}\pi r \left[(\rho_0 + \lambda a)^2 - (\rho_0 + \lambda a) \frac{7}{4}\lambda r + \frac{3}{4}\lambda^2 r^2 \right] dr.$$

Integrating,

$$p = A - \frac{4}{3}\pi \left[(\rho_0 + \lambda a)^2 \frac{1}{2}r^2 - (\rho_0 + \lambda a) \frac{7}{12}\lambda r^3 + \frac{3}{16}\lambda^2 r^4 \right]$$

Let p_0 be the pressure at the centre.

$$p = p_0 \text{ when } r = 0; \quad A = p_0.$$

$$\therefore p = p_0 - \frac{4}{3}\pi \left[(\rho_0 + \lambda a)^2 \frac{1}{2}r^2 - (\rho_0 + \lambda a) \frac{7}{12}\lambda r^3 + \frac{3}{16}\lambda^2 r^4 \right].$$

But when $r = a, p = 0$.

$$\begin{aligned} \therefore p_0 &= \frac{4}{3}\pi \left[(\rho_0 + \lambda a)^2 \frac{1}{2}a^2 - (\rho_0 + \lambda a) \frac{7}{12}\lambda a^3 + \frac{3}{16}\lambda^2 a^4 \right] \\ &= \frac{4}{3}\pi \left[\{\rho_0 + 4(\rho - \rho_0)\} \frac{1}{2}a^2 - \{(\rho_0 + 4(\rho - \rho_0)) \frac{7}{12}\lambda a^3 + \frac{3}{16}a^4 \cdot 16(\rho - \rho_0)\} \right] \\ &= \frac{4}{3}\pi^2 a \cdot \frac{1}{6} [(4\rho - 3\rho_0)^2 - 14(\rho - \rho_0)(4\rho - 3\rho_0) + 18(\rho - \rho_0)^2] \\ &= \frac{2}{9}\pi a^2 [10\rho^2 - 10\rho\rho_0 + 3\rho_0^2] \\ &= \frac{2}{9}\pi a^2 [10\rho(\rho - \rho_0) + 3\rho_0^2]. \end{aligned}$$

$$v^{\mu} \nabla_{\mu} \psi + c' \mu \psi' \}$$

Pressure of Rotating Fluids

6.14 : A liquid revolve with a uniform angular velocity ω about the vertical axis, let the liquid such that there is no relative displacement of its particles, i.e., the rotation like a rigid body about the axis.

Let us consider a particle m at a radial distance r from the axis of rotation as the axis of z . Now considering a particle m at a radial distance r from the axis. This particle is moving in a circle of radius r with uniform tangential velocity ωr ; so the tangential acceleration due to rotation vanishes and the angular velocity $\omega^2 = r \left(\frac{d\theta}{dt} \right)^2$ acts along the inward drawn normal (i.e., towards the normal acc i.e. $r\omega^2$)

along the normal towards the axis being $m\omega^2 r$.

If we apply a force $m\omega^2 r$ away from the axis, the force due to rotation is balanced and the system is in static equilibrium.

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The components of this applied force $m\omega^2 r$ along the axis are

$$m\omega^2 r \cdot \frac{x}{r} = m\omega^2 x \text{ along } x\text{-axis};$$

$$m\omega^2 r \cdot \frac{y}{r} = m\omega^2 y \text{ along } y\text{-axis}.$$

If X, Y, Z be the external forces per unit mass acting on the fluid, then

along x -axis	along y -axis	along z -axis
$m\omega^2 x$	$m\omega^2 y$	$m\omega^2 r$

total external force along x -axis = $m\omega^2 x$,
total external force along y -axis = $m\omega^2 y$,
total external force along z -axis = $m\omega^2 r$.

∴ The pressure p at the point (x, y, z) is given by

$$dp = p[(X + \omega^2 x)dx + (Y - \omega^2 y)dy + (Z + \omega^2 r)dz]$$

$$= p[(X dx + Y dy + Z dz) + \omega^2(x dx + y dy) + \omega^2 r dz]$$

Which in general gives the pressure at a point of the rotating fluid.

Cor 1.

A mass of homogeneous liquid, contained in a vessel, revolves uniformly about a vertical axis, to determine the pressure at any point and the surfaces of the equal pressure.

In this case acceleration due to gravity is the only external force, the force due to rotation being $m\omega^2 r$ perpendicular to the axis of rotation here).

Thus force along x -axis = $m\omega^2 r \frac{x}{r} = m\omega^2 x$,

The force along y -axis = $m\omega^2 y$

and the force along z -axis = $-mg$.

∴ the differential equation for pressure is

$$dp = p[\omega^2 x dx + \omega^2 y dy - g dz].$$

$$\text{Integrating } p = p \left[\frac{1}{2}\omega^2(x^2 + y^2) - gz \right] + C. \quad \dots(1)$$

This gives pressure at any point.

Putting $p = \text{const.}$, the surfaces of equal pressure are of the form

$$\frac{1}{2}\omega^2(x^2 + y^2) - gz = \text{const.}$$

or

$$x^2 + y^2 = \frac{2g}{\omega^2}(z - c) \quad \dots(2)$$

which clearly represents paraboloids of revolution of latus rectum $\frac{2g}{\omega^2}$, where c is an arbitrary constant.

Cor 2. if the vessel is open at the top, i.e., the pressure at the highest point is same as the atmospheric pressure, then in (1), putting $p = \Pi$, the equation of the free surface is given by

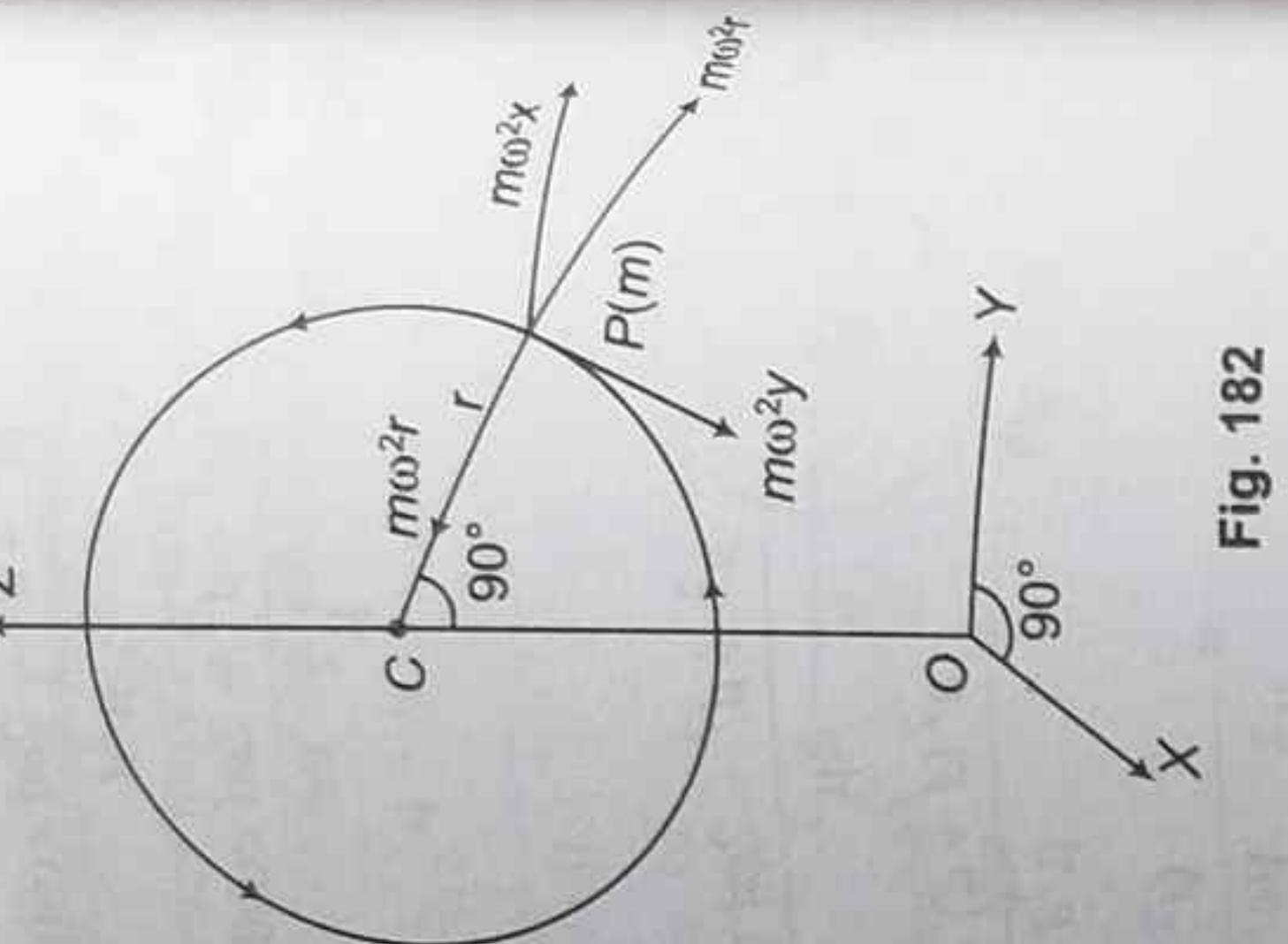


Fig. 182

Introducing the vessel is origin and the axis of rotation is vertical.

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$$\Pi = \rho \left[\frac{1}{2} \omega^2 (x^2 + y^2) - gz \right] + C$$

$$\omega^2 (x^2 + y^2) - 2gz + \frac{2C}{\rho} = \frac{2\Pi}{\rho}.$$

If constant C can be determined by the initial conditions. For example, if the constant at the top and be just filled with liquid, then $\Pi = 0$ and taking the closed point of the axis, i.e., when $x = 0, y = 0, z = 0, p = 0 \therefore C = 0$, then at the highest point of the axis, i.e., when $x = 0, y = 0, z = h, p = \rho \left[\frac{1}{2} \omega^2 (x^2 + y^2) - gh \right]$

and

Elastic Fluid Rotating with Constant Angular Velocity

6.15: Elastic fluid rotating with constant angular velocity, we have from Boyle's law,

$$p = kp \quad \text{or} \quad dp = k \, dp.$$

Also we have as usual for rotating heavy fluid

$$dp = \rho [\omega^2 (x \, dx + y \, dy) - g \, dz]$$

$$k \, dp = \rho [\omega^2 (x \, dx + y \, dy) - g \, dz]$$

$$k \frac{dp}{\rho} = \omega^2 (x \, dx + y \, dy) - g \, dz.$$

$$k \log \rho = \frac{1}{2} \omega^2 (x^2 + y^2) - gz + C. \quad \dots(1)$$

Integrating,

Putting $\rho = \text{const.}$, we find that here also surfaces of equal pressure (or equal density) are paraboloids of revolution. The constant of integration C is determined with the help of the given conditions.

Example 35. An open vessel containing liquid is made to revolve about a vertical axis with uniform angular velocity. Find the form of the vessel and its dimensions that it may be just emptied.

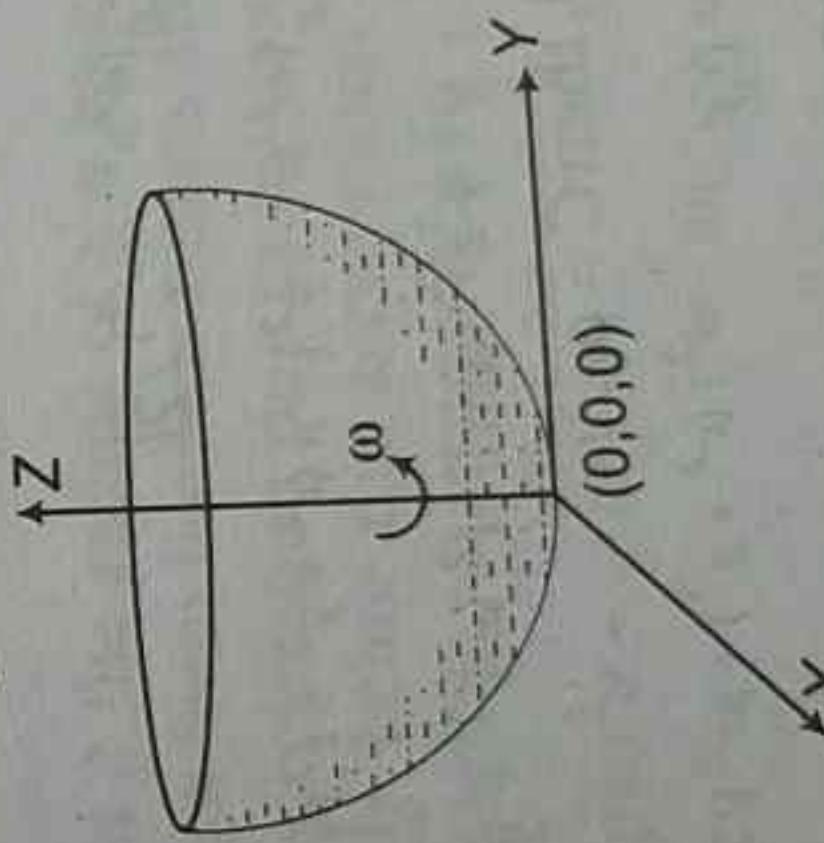
Sol. Let the liquid revolve with uniform angular velocity ω , then pressure at any point is given by

$$dp = \rho [(\omega^2 x \, dx + \omega^2 y \, dy) - g \, dz].$$

Integrating,

$$p = \rho \left[\frac{1}{2} \omega^2 (x^2 + y^2) - gz \right] + C. \quad \dots(2)$$

Let us take the lowest point of the vessel to be the origin. Then since the vessel is just emptied, there is no liquid or liquid pressure at the lowest point.



... (2)

where c is an

constant.

$\therefore p = 0$, when $x = y = z = 0$, or $C = 0$.

Fig. 184

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Hence (1) becomes $p = \rho \left[\frac{1}{2} \omega^2 (x^2 + y^2) - gz \right]$.

Again since the vessel is just emptied, the inner surface of the vessel coincides with the free surface of the liquid. So putting $p = 0$, the free surface of the liquid, the inner surface of the vessel is given by

$$\frac{1}{2} \omega^2 (x^2 + y^2) - gz = 0$$

or

$$x^2 + y^2 = \frac{2g}{\omega^2} z$$

which is a paraboloid of revolution of latus rectum $\frac{2g}{\omega^2}$.

This gives the required result.

Example 37. A given mass of air is contained within a closed air-tight cylinder with its vertical axis rotating in relative equilibrium about the axis of the cylinder. The pressure at the highest point of its axis is P , and the pressure at the highest point of its curved surface is p . Prove that, if the fluid were absolutely at rest, the pressure at the upper end of the cylinder would be $\frac{p - P}{\log p - \log P}$, where the weight of the air is taken into account.

$$\rho(x,y,z)$$

$$\omega$$

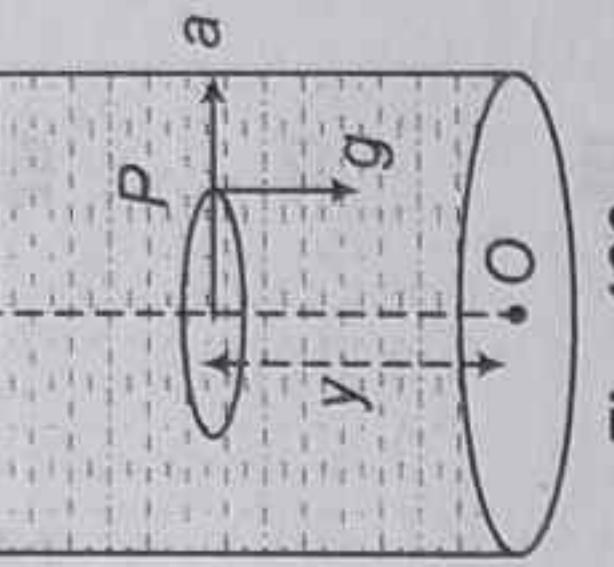


Fig. 186

Sol. Let h be the height and a the radius of the base of the cylinder. The point at a distance u from the axis rotates about the axis in a circle of radius u , the acceleration due to rotation being $\omega^2 u$ in the direction of u and g being the acceleration due to gravity acting downwards. Thus while rotating we have

$$dp = \rho(\omega^2 u - g) dz \quad \dots(1)$$

The air is an elastic fluid

$$\dots(1)$$

therefore

$$p = k\rho \text{ i.e., } dp = k d\rho.$$

∴

$$k dp = \rho(\omega^2 u du - g dz)$$

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$$\frac{dp}{k} = \frac{1}{k} (\omega^2 u du - g dz).$$

or

$$p = C e^{\left(\frac{1}{k} \left(\frac{\omega^2 u^2}{2} - gz \right) \right)}$$

Integrating, $\log p = \log C + \frac{1}{k} \left(\frac{\omega^2 u^2}{2} - gz \right)$

$$\rho = C e^{\left(\frac{\omega^2 u^2}{2k} - \frac{gz}{k} \right)} \quad \text{or} \quad p = k\rho = kC e^{\left(\frac{\omega^2 u^2}{2k} - \frac{gz}{k} \right)}$$

$\rho = C e^{\left(\frac{\omega^2 u^2}{2k} - \frac{gh}{k} \right)}$ or pressure at any point of the air, when it is revolving.

This gives pressure at any point of the axis $u = 0, z = h$ and there pressure is P .

For highest point of the curved surface, $u = a, z = h$ and there the pressure is given by

\therefore

For highest point of the curved surface, $u = a, z = h$ and there the pressure is given by

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For highest point of the curved surface, $u = a, z = h$ and there the pressure is given by

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$$\frac{p - P}{\log p - \log P} = \frac{2k^2 C e^{-gh/k} (e^{\omega^2 a^2 / 2k}) - 1}{\omega^2 a^2}$$

So

$$p = k C e^{\left(\frac{\omega^2 u^2}{2k} - \frac{gh}{k} \right)}$$

Now let us consider the air when it is at absolute rest; then the pressure p' at any point is given by (now $\omega = 0$)

$$dp' = -g\rho dz \quad \text{or} \quad k d\rho = -g\rho dz$$

$$\frac{d\rho}{\rho} = -\frac{g}{k} dz \quad \text{or} \quad \rho = A e^{-gz/k}.$$

or

\therefore

$$p' = k\rho - kAe^{-gz/k}.$$

This gives pressure at any point when the liquid is at rest.

Putting $z = h$, the pressure $p'E'$ at the upper end E of the axis is given by

$$p' = k A e^{-gh/k}.$$

But the mass of the air remains unchanged whether it revolves or not. (Mass when rotating = mass when at rest)

$$\therefore \int_{z=0}^h \int_{u=0}^a C e^{\left(\frac{\omega^2 u^2}{2k} - \frac{gz}{k} \right)} \cdot 2\pi u du dz = \int_{z=0}^h \int_{u=0}^a A e^{-gz/k} 2\pi u du dz$$

$$\text{or } 2\pi C \int_{z=0}^h e^{-gz/k} dz \int_{u=0}^a e^{\omega^2 u^2 / 2k} u du = 2\pi A \int_{z=0}^h e^{-gz/k} dz \int_{u=0}^a u du$$

$$\text{or } C \left[-\frac{k}{8} e^{-gz/k} \right]_0^h \left[\frac{k}{\omega^2} e^{\omega^2 u^2 / 2k} \right]_0^a = A \left[-\frac{k}{8} e^{-gz/k} \right]_0^h \left[\frac{u^2}{2} \right]_0^a$$

or

$$\frac{Ch}{\omega^2} [e^{\omega^2 a^2 / 2k} - 1] = A \frac{a^2}{2}$$

or

$$A = \frac{2Ch}{\omega^2 a^2} (e^{\omega^2 a^2 / 2k} - 1) = \frac{1}{k} e^{gh/k} \frac{p - P}{\log p - \log P}$$

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Putting this value of A in (4),

pressure at the upper end of the axis when air is at rest = $\frac{p - P}{\log p - \log P}$.

p_0' = pressure at

This proves the result.

Example 38. Fluid is contained within a circular tube of radius a in a vertical plane

or not.

from (3)

$$w^2 = \frac{g}{a} \sin^2 \frac{\theta}{4}$$

$$w = \sqrt{\frac{g}{a}} \sec \frac{\theta}{4}$$

Example 39. A conical vessel, of height h and vertical angle 2α , contains water whose volume is one half that of the cone, of the vessel and the contained water revolve with uniform angular velocity w , and no water overflow, show that w must not be greater than $\sqrt{\frac{2g}{3h}} \cot \alpha$. [CU-2004H]

Sol. Let V be the vertex of the paraboloid of free surface during rotation about the vertical axis AZ .

Let O be the centre of circular base of the cone and P be a point on the right of this circular base of the cone thus

$$OP^2 = \frac{2g}{w^2} OV \quad \dots(1)$$

$$\text{In } \triangle OAP, \tan \alpha = \frac{r}{h} = \frac{OP}{h} \Rightarrow OP = h \tan \alpha \quad \dots(2)$$

Hence from (1), we have

$$(h \tan \alpha)^2 = \frac{2g}{w^2} \cdot OV$$

$$\Rightarrow OV = \frac{h^2 w^2 \tan^2 \alpha}{2g} \quad \dots(3)$$

If no water is to overflow the volume of empty space PVQ cannot be greater than half of the volume of the cone.

$$\text{or} \quad \frac{1}{2} \pi h^2 \tan^2 \alpha \cdot OV > \frac{\pi}{6} h^3 \tan^2 \alpha.$$

$$\text{or} \quad OV > \frac{h}{3}$$

$$\text{or} \quad \frac{w^2 h^2 \tan^2 \alpha}{2g} > \frac{h}{3}$$

$$\text{or} \quad w^2 > \frac{2gh \cot^2 \alpha}{3h^2}$$

$$\text{or} \quad w > \sqrt{\frac{2g}{3h}} \cot \alpha.$$

using (3)

Example 40. A right circular cylinder, open at the top, is filled with water and whole of it revolves with angular velocity w about the axis. If not more than half the water is split prove that the thrust on the base is $\pi \rho g a^2 h \left(1 - \frac{a^2 w^2}{4gh} \right)$, where h is the height, a the radius of the base of the cylinder and ρ is the density of the water.

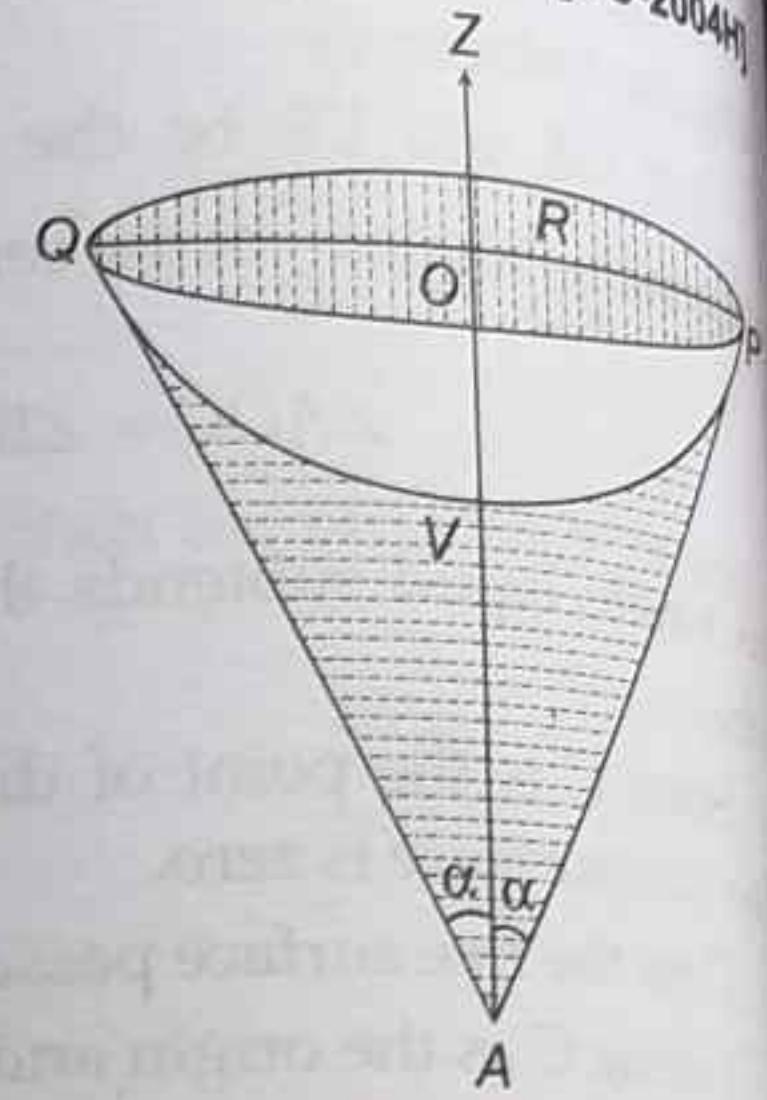


Fig. 188

Sol. Let $ABCD$ be a right circular cylinder with height h and radius a . The centre O of the base is twice as the origin w the axis OZ as the axis of rotation and two mutually perpendicular lines OX and OY as the plane base as the x -axis & y -axis respectively.

Let V be the vertex of the paraboloid of revolution of the latus rectum $\frac{2g}{w^2}$. Clearly free surface of the liquid during rotation about the vertical axis OZ is a paraboloid.

The pressure at any point $P(x, y)$ on the plane base is given by

$$p = \frac{1}{2} \rho w^2 r^2 - \rho g z + k \quad \dots(1)$$

At $V, r = 0, z = 0, p = 0$, then from (1) $k = 0$

$$p = \frac{1}{2} \rho w^2 r^2 - \rho g z \quad \dots(2)$$

At B , we have $r = a, z = h - OA, p = 0$, hence from (2), we get

$$0 = \frac{1}{2} \rho w^2 a^2 - \rho g (h - OV)$$

$$\rho g OV = \rho gh - \frac{1}{2} \rho a^2 w^2 \quad \dots(3)$$

If P is a point on the base as p pressure there, $OP = r$, then

$$\begin{aligned} p &= \frac{1}{2} \rho w^2 r^2 - \rho g (-OV) \\ &= \frac{1}{2} \rho w^2 r^2 + \rho gh - \frac{1}{2} \rho a^2 w^2 \quad \text{using (3)} \\ p &= \rho gh - \frac{1}{2} \rho w^2 (a^2 - r^2) \quad \dots(4) \end{aligned}$$

Hence the required thrust on the plane base

$$\begin{aligned} &= \int_{\theta=0}^a \int_{\theta=0}^{2\pi} pr d\theta dr \\ &= \int_0^a p r 2\pi dr = \int_0^a \pi [2\rho gh - \rho w^2 (a^2 - r^2)] r dr \quad \text{using (4)} \\ &= \pi \left[2\rho gh \frac{r^2}{2} - \rho w^2 \left(a^2 \frac{r^2}{2} - \frac{r^4}{4} \right) \right]_0^a \\ &= \pi \rho g a^2 h \left(1 - \frac{a^2 w^2}{4gh} \right) \end{aligned}$$

Also, volume of water split = volume of the paraboloid.

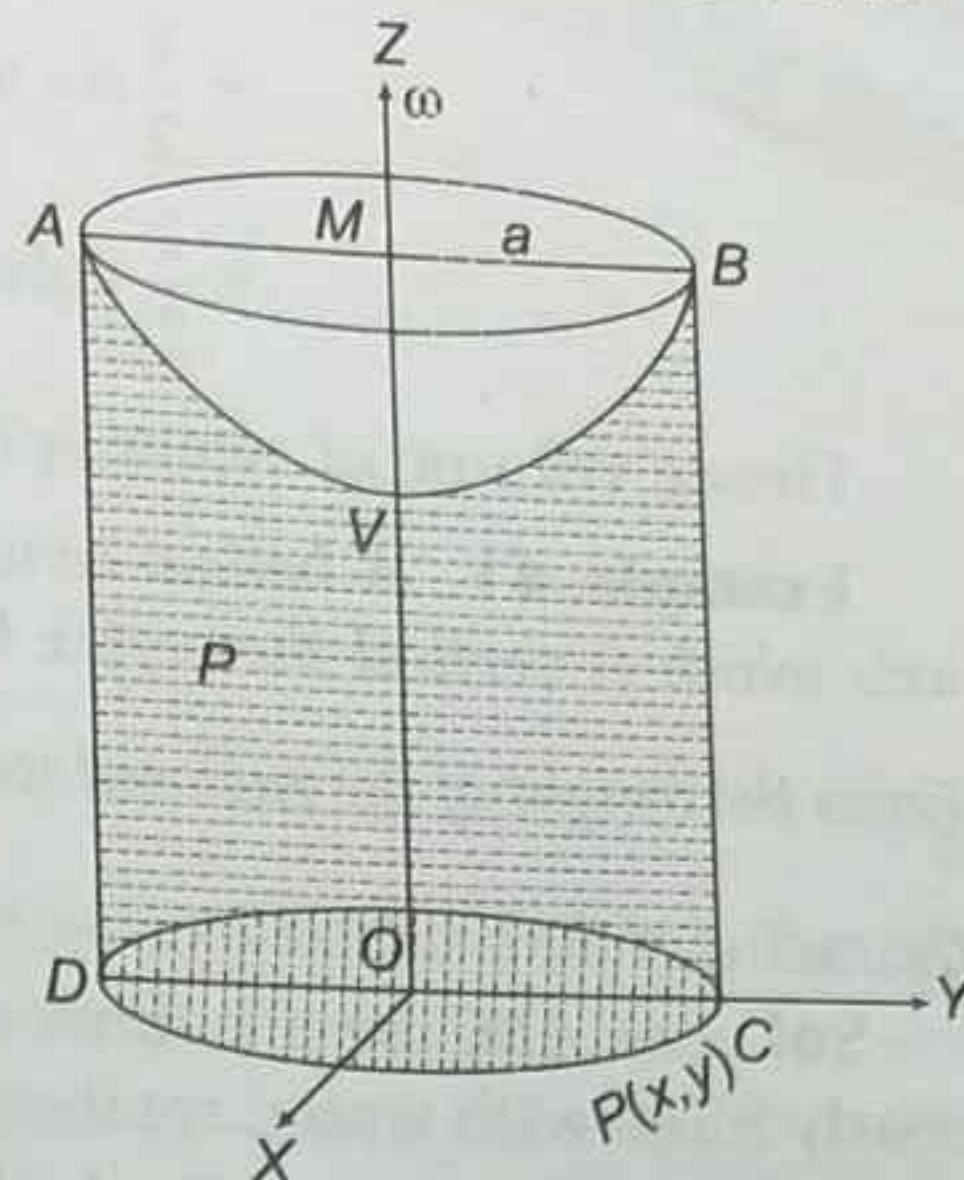


Fig. 189

$$= \frac{1}{2} \text{ area } (AB \times MV) < \frac{1}{2} \text{ area } (AB \times MO)$$

$$= \frac{1}{2} \text{ volume of the cylinder.}$$

Hence volume of water split is less than half of the volume of the cylinder.

Example 41. A hollow cone very nearly filled with water, rotates uniformly about its axis which is vertical the vertex being uppermost. If the pressure or the base be equal to six times the weight of the enclosed water, prove that the angular velocity is $\sqrt{\frac{4g}{a} \cos \alpha}$, where a is

the radius of the base and 2α semi-vertical angle of the cone.

Sol. Let ABC be a hollow cone of height $OB = h$ and semi-vertical angle α very nearly filled with water, rotates uniformly about its axis OC with angular velocity ω .

Let $OA = OB = a$ & $OC = h$, then the pressure at any point is given by

$$dp = \rho [w^2 r dr - g dz]$$

Integrating, we get

$$p = \rho \left[\frac{w^2 r^2}{2} - gz \right] + C$$

Now, at $C, p = 0, r = 0$ & $z = h, \Rightarrow C = \rho gh$.

$$p = \rho \left(\frac{1}{2} w^2 r^2 - gz \right) + \rho gh.$$

On the circular base of the cone, $z = 0$ so pressure at a point on the circular base is given by

$$(1) \quad p = \rho \left(\frac{1}{2} w^2 r^2 + gh \right)$$

Thus the total thrust on plane base is given by

$$\begin{aligned} \pi &= \int_{r=0}^a \int_{\theta=0}^{2\pi} \rho \left(\frac{1}{2} w^2 r^2 + gh \right) r dr d\theta \\ &= 2\pi \rho \int_0^a \left(\frac{w^2 r^2}{2} + ghr \right) dr \\ &= \pi \rho a^2 \left(\frac{w^2 a^4}{4} + \frac{gha^2}{2} \right) \\ &= \pi \rho a^2 \left(\frac{w^2 a^2}{4} + gh \right) \end{aligned}$$

Now the weight of the enclosed water is given by

$$W = \frac{1}{3} \pi a^2 h g \rho.$$

But from the question, $T = 6W$

$$\therefore \pi \rho a^2 \left(\frac{w^2 a^2}{4} + gh \right) = 6 \cdot \frac{1}{3} \pi a^2 h g \rho$$

$$\frac{w^2 a^2}{4} + gh = 2gh.$$

or

$$\frac{w^2 a^2}{4} = gh$$

$$w^2 a^2 = 4gh$$

$$w^2 = \frac{4g}{a} \cdot \frac{h}{a} = \frac{4g \cot \alpha}{a}$$

$$w = \sqrt{\frac{4g \cot \alpha}{2}}.$$

Example 42. A narrow horizontal tube BC has two open vertical branches BA and CD. Water is poured into the continuous tube so that h is the height of water in each of the two vertical branches. The whole tube is set rotating about a vertical axis through a point O in BC. Show that in a state of relative equilibrium the difference of level in the two branches is

$$\frac{\omega^2}{2g} (OB^2 - OC^2)$$

where ω is the angular velocity [no liquid passes out of the vertical branches].

Sol. When rotated about a vertical axis through O, the free surface will be a parabola EVF whose equation is

$$y = \frac{\omega^2}{2g} x^2.$$

x-coordinate of E = $-OB$,

x-coordinate of F = $+OC$.

$\therefore y_1 = y$ coordinate of E

$$= \frac{\omega^2}{2g} OB^2,$$

$$y_2 = y$$
-coordinate of F = $\frac{\omega^2}{2g} OC^2.$

$$\therefore y_1 - y_2 = \frac{\omega^2}{2g} (OB^2 - OC^2)$$

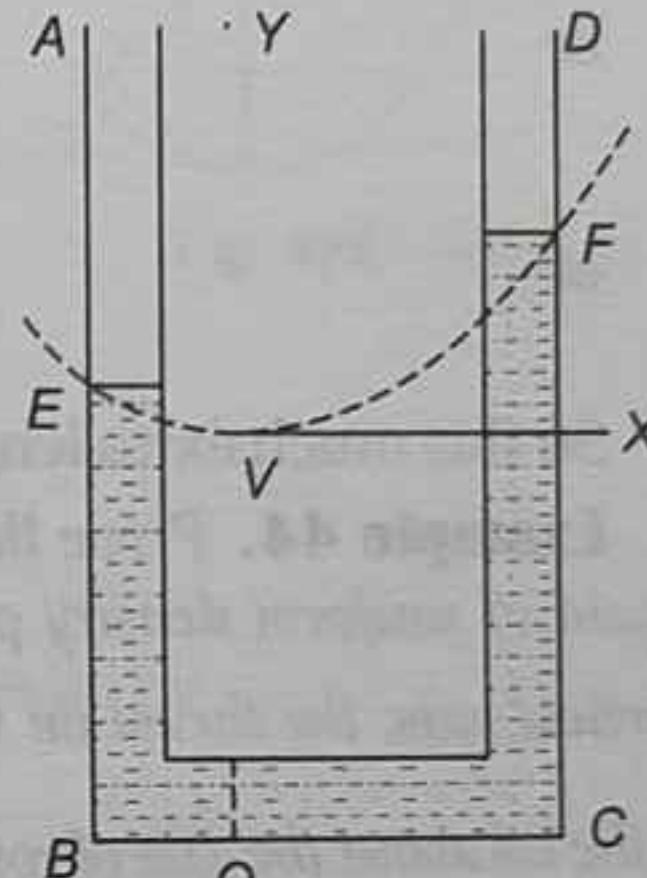


Fig. 190

This gives the difference of level in the two branches.

Example 43. A circular cylinder of radius a , is floating freely in water with its axis vertical. At first the water is at rest and it is then made to rotate about an axis containing with the axis of the cylinder with angular velocity ω . Show that in the second case an extra length $\frac{\omega^2 a^2}{4g}$ of the surface of the cylinder is wetted.

Sol. Let h be the height of the cylinder of density ρ immersed in the liquid of density ρ' to a height h' . Then for floating in equilibrium at rest,

$$\pi a^2 h \rho = \pi a^2 h' \rho',$$

i.e.

$$h' = \left(\frac{\rho}{\rho'} \right) h. \quad \dots(1)$$

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When the liquid rotates about the axis of the cylinder as axis of rotation, the free surface is a paraboloid of revolution, the equation of whose section referred to its vertex C as origin is

$$y = \frac{\omega^2}{2g} x^2$$

or

$$NC = \frac{\omega^2}{2g} \cdot AN^2$$

[for the pt. A]
[as $AN = a$.]

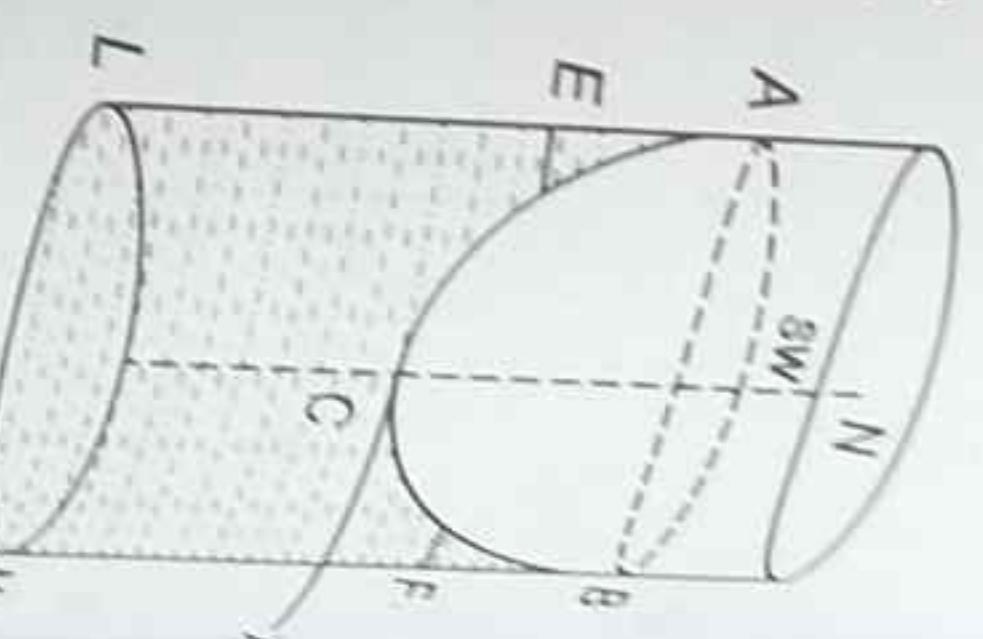


Fig. 191

Let

$AE = x$, then for floating

$a^2 h \rho$ = weight of the cylinder

= weight of the water displaced

= weight of liquid $ALMB$ - weight of liquid ACB

$$= \left[\pi a^2 (h' + x) + \frac{1}{2} \pi a^2 \cdot NC \right] \rho'$$

$$\text{or } h\rho = \left(h' + x - \frac{1}{2} \frac{\omega^2 a^2}{2g} \right) \rho' = \left(h' + x - \frac{\omega^2 a^2}{4g} \right) \frac{h}{h'} \rho \quad [\text{from } (1)]$$

$$\text{or } \left(x - \frac{\omega^2 a^2}{4g} \right) \frac{h}{h'} \rho = 0$$

or

$$x = \frac{\omega^2 a^2}{4g}$$

So this much extra length of the surface of the cylinder is wetted.

Example 44 *Determine the extra length of the surface of a cylinder of diameter 1 m which is just full of water when it rotates with an angular velocity of 10 rad/s .*

The

higher

point

In