

# **Centre of Pressure of a Plane Area**

## § 4.1 : How to find the depth of C.P of the plane area :

[BNMU-15H]

Let a plane area  $S$  be immersed vertically in a liquid of density  $\rho$ . Considering an elementary area  $ds$  in the plane area  $S$  at a depth  $z$  below the free surface.

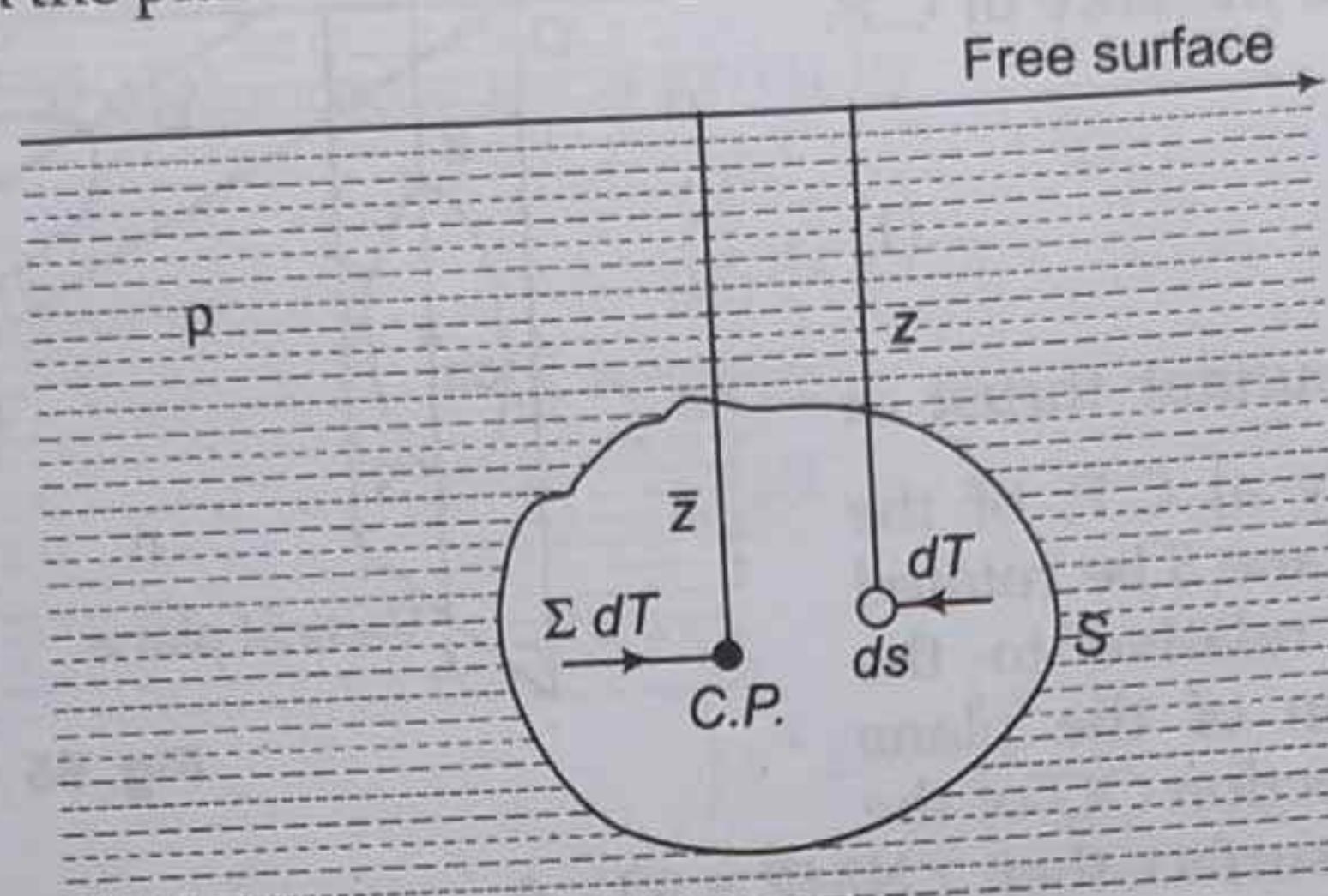


Fig. 64

Let  $dT$  be the elementary thrust on the elementary area  $ds$ , then

$$dT = \text{area} \times \text{pressure at its C.G.}$$

$$= ds \times z\rho g$$

$$dT = \rho g z ds.$$

Let  $\bar{z}$  be the depth of C.P. of the area  $S$ , then the resultant thrust  $\Sigma dT = \int dT$  will act normally at the C.P. of the plane area  $S$ .

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By varignan's theorem in statics, the algebraic sum of the moments of the resultant thrust acting on the plane lamina about a line is same as the moment of the resultant thrust about the same line i.e., the sum of the moments about the free surface is equal to the moment of the different elements of the area  $S$  about the free surface.

resultant  $\Sigma dT$  acting at the C.P. about the free surface.

$$\begin{aligned}\Sigma z \, dT &= \bar{z} \Sigma dT \\ \therefore \bar{z} &= \frac{\Sigma dT}{\Sigma z \, dT} \\ &\Rightarrow \bar{z} = \frac{z \, dT}{dT} \\ &= \frac{\int z \rho g \, z \, ds}{\int z \, ds} \\ &= \frac{\int \rho z^2 \, ds}{\int \rho z \, ds}\end{aligned}$$

is the required formula for the depth of C.P. of the plane area when liquid is heterogeneous.

**Cor:** If the liquid is homogeneous then  $\rho$  is constant

$$\therefore \bar{z} = \frac{\int z^2 \, ds}{\int z \, ds}$$

### § 4.2 : To show that the position of the C.P. of the area is independent of the rotation relative to vertical position about the line of intersection of the plane and free surface

Let a plane area  $S$  be immersed vertically in a liquid of density  $\rho$ .

Let  $ds$  be the elementary area in the vertical plane at a depth  $z$  below the free surface, then from the previous theorem, the distance of C.P. is given by

$$\bar{z} = \frac{\int z \, dT}{\int dT} \quad \dots(1)$$

where  $\int dT$  is the resultant thrust of liquid acting normally at C.P. of the plane. Now, let the plane  $S$  be rotated through an angle  $\theta$  relative to the initial vertical position of the plane about the line of intersection  $OQ$  of the vertical plane and free surface then  $\angle NMP = 90 - \theta$ .

Thus the depth of the element is  $z \sin (90 - \theta) = z \cos \theta$ .

The thrust on the elementary area  $ds$  is given by

$$dT = ds \times z \cos \theta \rho g = \rho g z \cos \theta \, ds$$

The depth of C.P. of the plane area =  $\bar{z} \cos \theta$ .

By varignon's theorem, the sum of moments of all thrusts acting at C.P. about the free surface is equal to the moment of the resultant thrust acting at C.P. about the same surface.

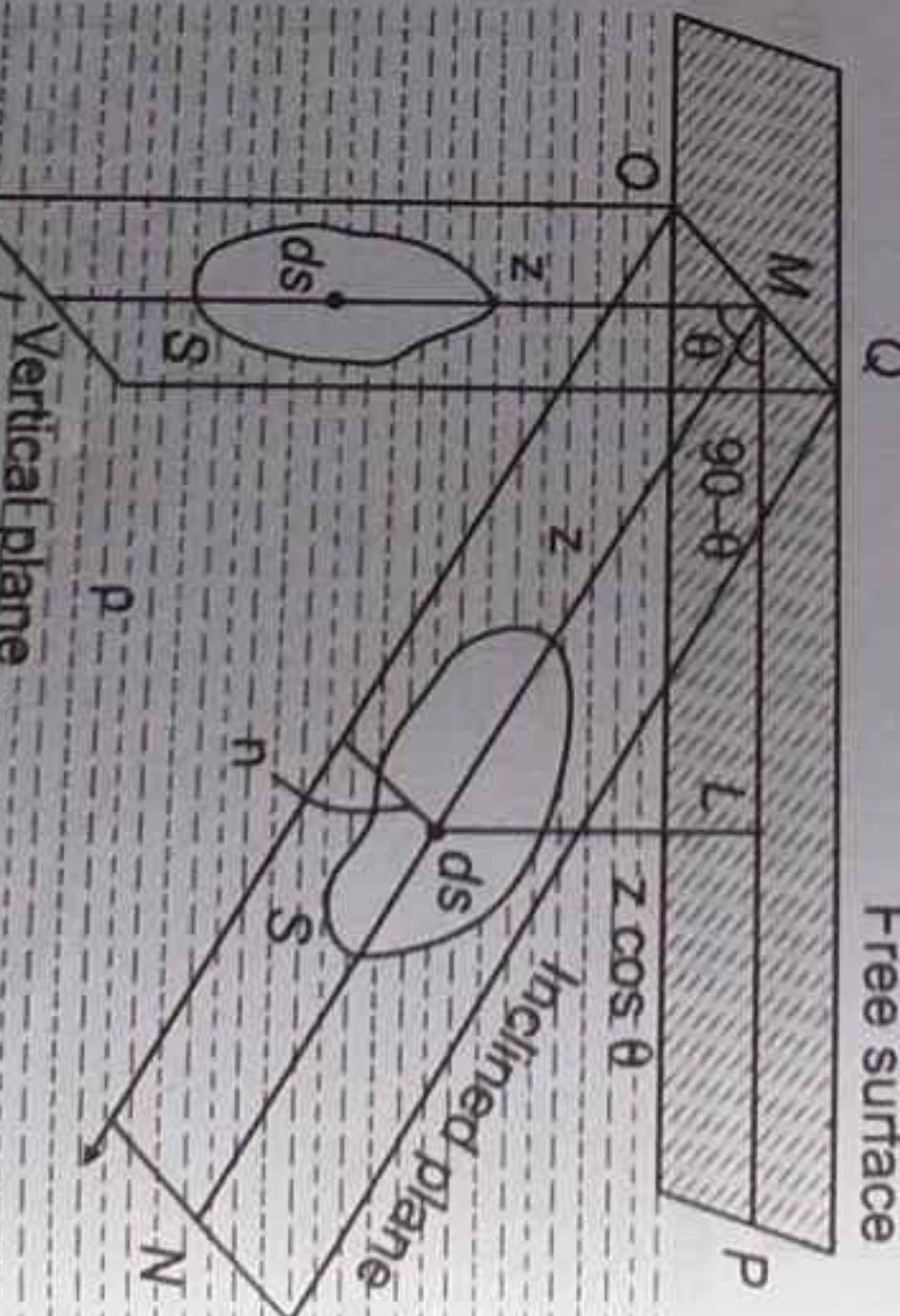


Fig. 65

$$\bar{x} = \frac{3}{2} \frac{2a+b}{3a^2 + 3ab + b^2}$$

$$= \frac{\int_a^{a+b} x dx}{\int_a^{a+b} x^2 dx} = \frac{\left[ \frac{x^2}{2} \right]_a^{a+b}}{\left[ \frac{x^3}{3} \right]_a^{a+b}}$$

$$\bar{x} = \frac{\int_a^b x dt}{\int_a^b dt}$$

The depth of C.P. of the rectangle is given by

$$= \rho g l dx$$

$$dT = l dx \rho g x$$

Thrust on the elementary strip PQ is given by

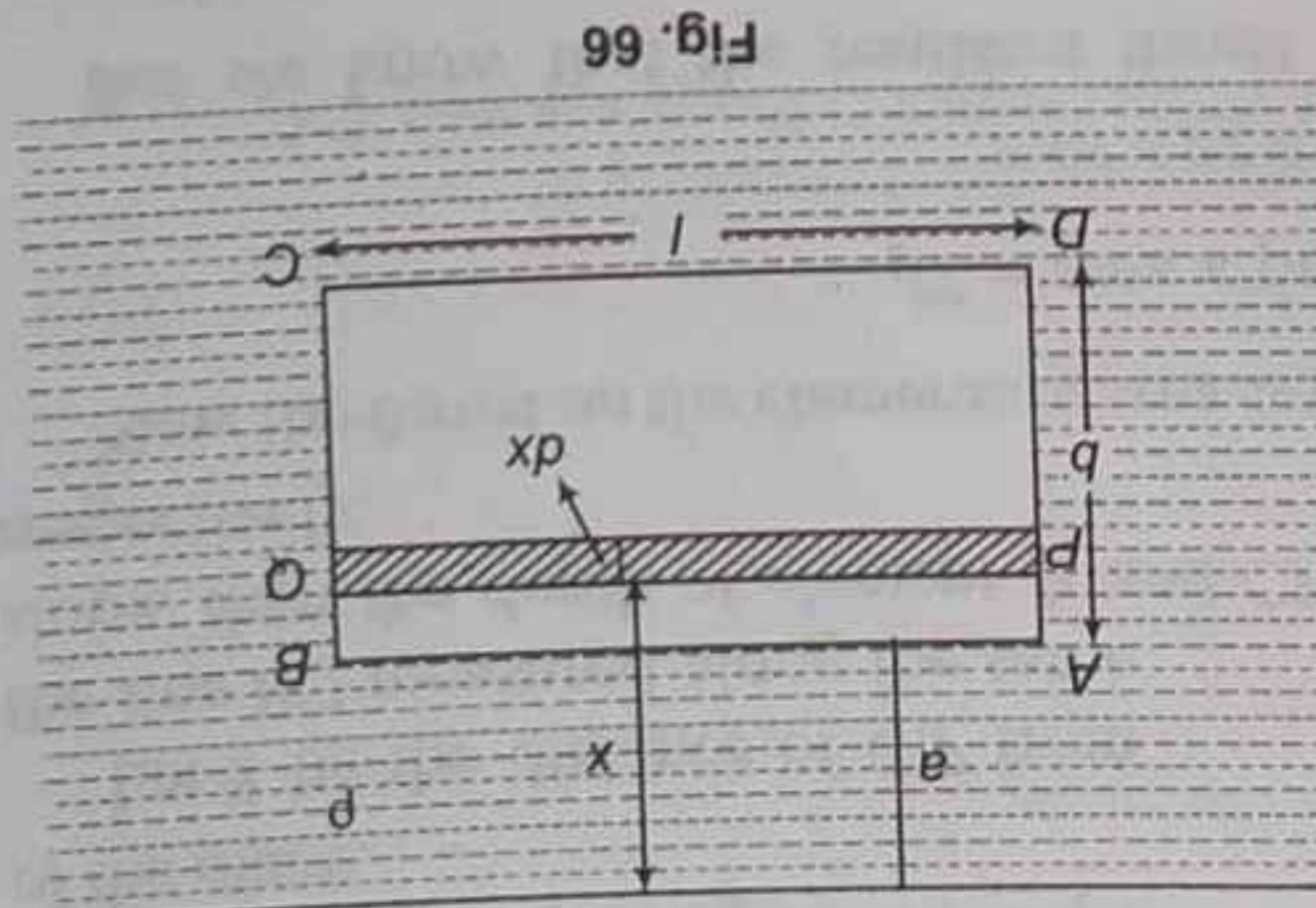


Fig. 66

**Example 1 :** Find the depth of C.P. of the rectangular Lamina whose two sides are free surface, the position of C.P. of the plane remains unaltered.

Thus by the rotation of the plane area about the line of intersection of the plane and horizontal below free surface of liquid.

$$z = \int \rho z ds \quad \text{which is independent of } \theta \text{ (rotation)}$$

$$z \cos^2 \theta \int \rho z ds = \cos^2 \theta \int \rho n^2 ds$$

$$z \cos \theta \int \rho g z \cos \theta ds = \int z \cos \theta \rho g z \cos \theta ds$$

$$z \cos \theta \int dt = \int z \cos \theta dt$$

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**§ 4.3 : The depth of C.P of a plane area immersed in a liquid is greater than the depth of C.G of the plane area**

**Sol.** Let ABCD be the free surface of liquid of density  $\rho$  and a plane area  $S$  be immersed in this liquid in such a way that its plane makes a constant angle  $\theta$  with the horizontal free surface. Let the plane area  $S$  intersect the free surface in a straight line PQ. Considering an elementary area  $ds$  on the plane S at a distance  $h$  from the line PQ measured along be the slope of the plane.

Let  $\bar{z}$  be the distance of C.P from the line PQ measured along the same slope, then the depth of element  $ds$  and CP below the surface are  $h \sin\theta$  and  $\bar{z} \sin\theta$  respectively.

Now the thrust on the elementary area  $ds$  is given by

$$dT = \text{area} \times \text{pressure at its C.G} \\ = ds \times \rho g h \sin\theta \quad \dots(1)$$

But we know that the resultant thrust  $\int dT$  of liquid on the plane are acts normally through C.P, so taking moments of all the thrusts acting on the plane about the line PQ, we have

$$\bar{z} \int dT = \int h dT \quad (\text{by Varignon's theorem of statics})$$

$$\bar{z} = \frac{\int h dT}{\int dT} = \frac{\int h \cdot \rho g h \sin\theta ds}{\int \rho g h \sin\theta ds} = \frac{\int h^2 ds}{\int h ds} \quad \dots(2)$$

This is the required formula for the distance of C.P of a plane area from the free surface along the slope of the plane.

Since the R.H.S of (2) is independent of  $\theta$ , hence the inclination of the plane does not effect the depth of C.P.

From eqn. (1) & (2), we have

$$\bar{z} = \frac{\int h^2 ds}{\int h ds} \quad \dots(3)$$

Here,  $\int h^2 ds$  = the moment of inertia of the whole plane area about the line PQ =  $S k^2$ ,

where  $S$  is the area of the whole plane area about the line PQ =  $S k^2$ ,

Also,

$$\int h ds = \text{moment of the plane area about the line PQ} \\ = Sh,$$

where  $\bar{h}$  is the distance of the C.G. of the whole plane along the slope.

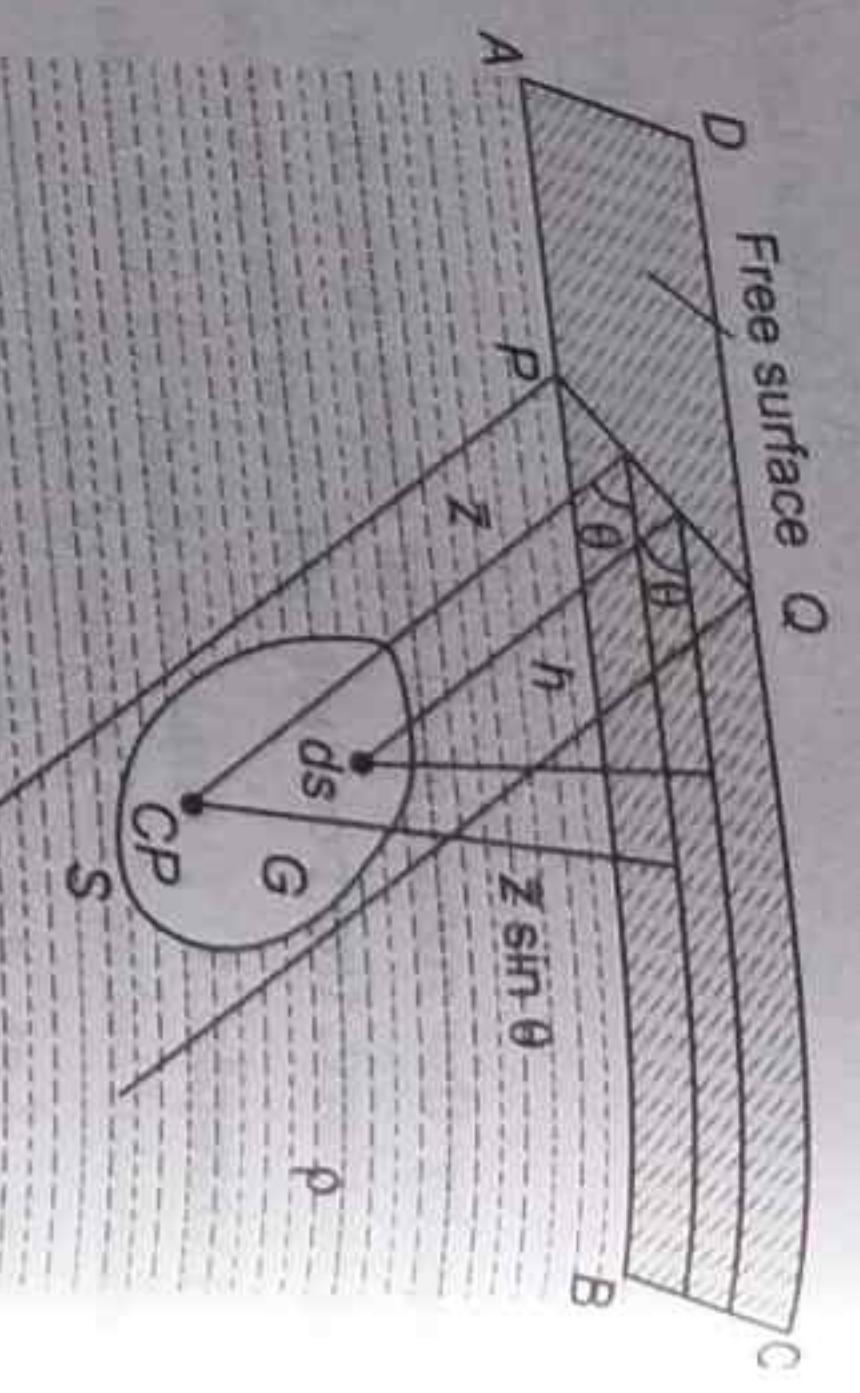


Fig. 67

Now from eqn. (3) and other results,

$$\bar{z} = \frac{Sk^2}{Sh} = \frac{k^2}{h} \quad \dots(4)$$

Let  $k'$  be the radius of gyration of the plane area about an axis through its C.G and parallel to the line  $PQ$ , then by the theorem of parallel axes, we have

$$Sk^2 = Sk'^2 + Sh\bar{z}^2 \quad \dots(5)$$

$$k^2 = k'^2 + \bar{h}^2$$

From (4) & (5) we get

$$\begin{aligned} \bar{z} &= k'^2 + \bar{h}^2 / \bar{h} \\ \bar{z} &= \bar{h} + k'^2 / \bar{h} \\ \bar{z} - \bar{h} &= k'^2 / \bar{h} > 0 \\ \bar{z} - \bar{h} &> 0 \\ \Rightarrow \quad \bar{z} &> \bar{h} \end{aligned} \quad \dots(6)$$

$\Rightarrow$  Distance of C.P > Distance of C.G.

**Cor : (i)** Effect of further immersion of the plane :

$$\text{From (6)} \quad \bar{z} - \bar{h} = \frac{k'^2}{\bar{h}} > 0$$

$$\begin{aligned} \Rightarrow \quad \frac{d(\bar{z} - \bar{h})}{d\bar{h}} &= \frac{k'^2}{\bar{h}^2} < 0 \\ \Rightarrow \quad \frac{d(\bar{z} - \bar{h})}{d\bar{h}} &< 0 \end{aligned}$$

$\Rightarrow$  Distance between C.P and C.G is decreasing with increase of  $\bar{h}$ .

Since the C.G of the lamina is a fixed point, hence C.P approaches C.G i.e. at a great depth C.P will coincide with C.G

**Cor : (ii)** From (6)

$$\bar{z} - \bar{h} = \frac{k'^2}{\bar{h}} > 0$$

Differentiating with respect to  $t$ , we get

$$\frac{d(\bar{z} - \bar{h})}{dt} = -\frac{k'^2}{\bar{h}^2} \times \frac{d\bar{h}}{dt} = \left(-k^2, \frac{d\bar{h}}{dt}\right) \cdot \frac{1}{\bar{h}^2}$$

This shows that the centre of pressure approaches the centre of gravity through the centre of gravity with a velocity which is inversely proportional to the square of the depth of its centre of gravity.

**Example 2.** Prove that the ...

**Example 3.** Find the depth of C.P of triangular lamina immersed in a liquid with one side on the surface of the liquid.

**Sol.** Let ABC be a triangle immersed in a liquid in such a way that the side AB be in the surface of the liquid. Let depth of C below the surface of the liquid be  $\gamma$  then the depth C.P is given by

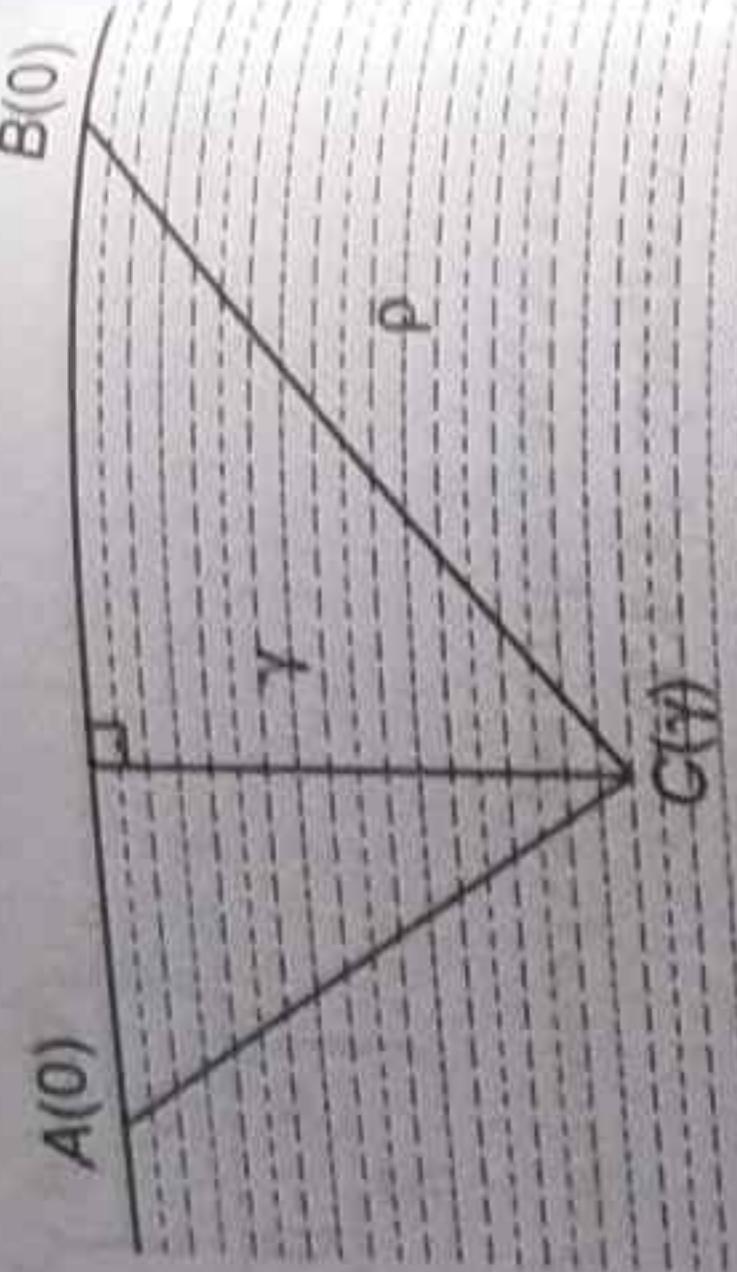


Fig. 70

$$\bar{z} = \frac{\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\beta+\gamma}{2}\right)^2 + \left(\frac{\gamma+\alpha}{2}\right)^2}{\left(\frac{\alpha+\beta}{2}\right) + \left(\frac{\beta+\gamma}{2}\right) + \left(\frac{\gamma+\alpha}{2}\right)}$$

$$= \frac{0^2 + \left(0 + \frac{\gamma}{2}\right)^2 + \left(\frac{\gamma+0}{2}\right)^2}{0 + \left(\frac{0+\gamma}{2}\right) + \left(\frac{\gamma+0}{2}\right)}$$

i.e.,

$$\bar{z} = \frac{\gamma}{2}$$

The depth of C.P of the triangle ABC is half of the depth of the vertex C.

#### **Second Method (The method of first principle)**

**Example 4.** Find the C.P. of triangular lamina immersed in a liquid with vertex in the surface and base horizontal.  
[TMBU-2013H, RU-2003H.]

**Sol.** Let ABC be the triangular lamina immersed in a liquid, with vertex A in the surface and base BC horizontal.

Let  $AD = r$  be the median taken as the  $x$ -axis and  $A$  as the origin. Let the median be inclined at angle  $\theta$  with the horizontal base. Let  $AE = c$  be the depth of the base  $BC$  below the surface. Divide the triangular lamina into elementary strips parallel to the free surface. Consider an elementary strip  $PQRS$  of width  $dx$  at a distance  $x$  from the origin  $A$  measured along the  $x$ -axis  $ADX$ .

Since the median  $AD$  intersects each strip at their middle points, hence the C.P. will be on the median  $AD$  i.e., on the  $x$ -axis.

The thrust on the elementary strip  $PQRS$  is given by

$$\begin{aligned} dT &= PQ \times dx \sin \theta x \sin \theta \rho g \\ &= \rho g PQ \cdot x \sin^2 \theta dx \end{aligned}$$

Let  $(\bar{x}, \bar{y})$  be the coordinates of C.P. then  $\bar{y} = 0$

$$\bar{x} = \frac{\int x \, dT}{\int dT}$$

$$\begin{aligned} &= \frac{\int_0^r x \rho g PQ x \sin^2 \theta dx}{\int_0^r \rho g PQ x \sin^2 \theta dx} \\ &= \frac{\int_0^r PQ x^2 \sin^2 \theta dx}{\int_0^r PQ \cdot x \sin^2 \theta d\theta}, \quad \text{where } \frac{PQ}{AM} = \frac{BC}{AD} PQ = \frac{ax}{r} \\ &\Rightarrow \bar{x} = \frac{\int_0^r x^3 dx}{\int_0^r x^2 dx} = \frac{3}{4}r \end{aligned}$$

$\therefore \bar{x} = \frac{3r}{4}$  = the distance of the C.P. from  $A$  along the median  $AD$

Hence the depth of C.P. below the free surface =  $(3/4) r \sin \theta = (3/4) AD \sin \theta$

**Example 5.** Find the centre of pressure of triangular area immersed in a liquid with one side in the surface (from the first principle).

[TMBU-2007(H), 2009H, 2011(H), 2015(H), RU-03H, BNMU-15H]

**Sol.** Let  $ABC$  be the triangular lamina immersed in a liquid of density  $\rho$  in such a way that its base  $BC$  be in the free surface of the liquid and vertex  $A$  be at a depth  $c$ .

Let  $AD = r$  be the median through  $A$  making an angle  $\theta$  with  $BC$ . Taking  $D$  as the origin and  $DAX$  along the  $x$ -axis. By the properties of C.P. it lies on the median  $DA$ .

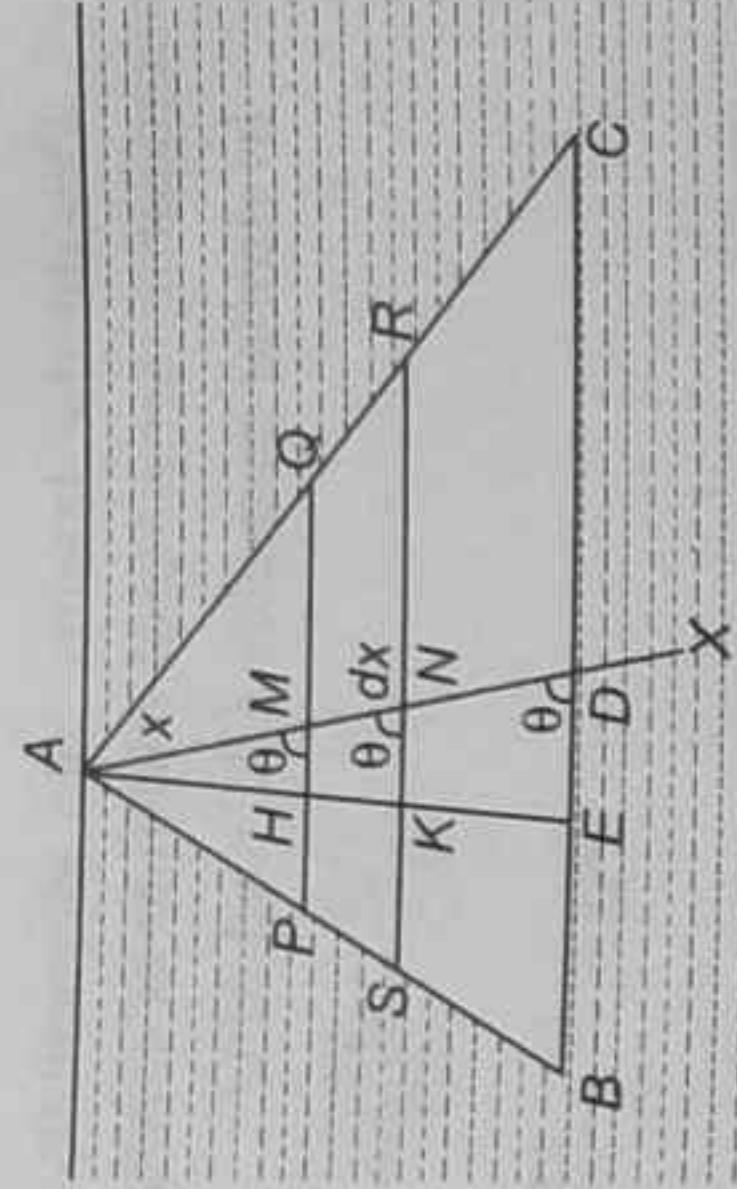


Fig. 71

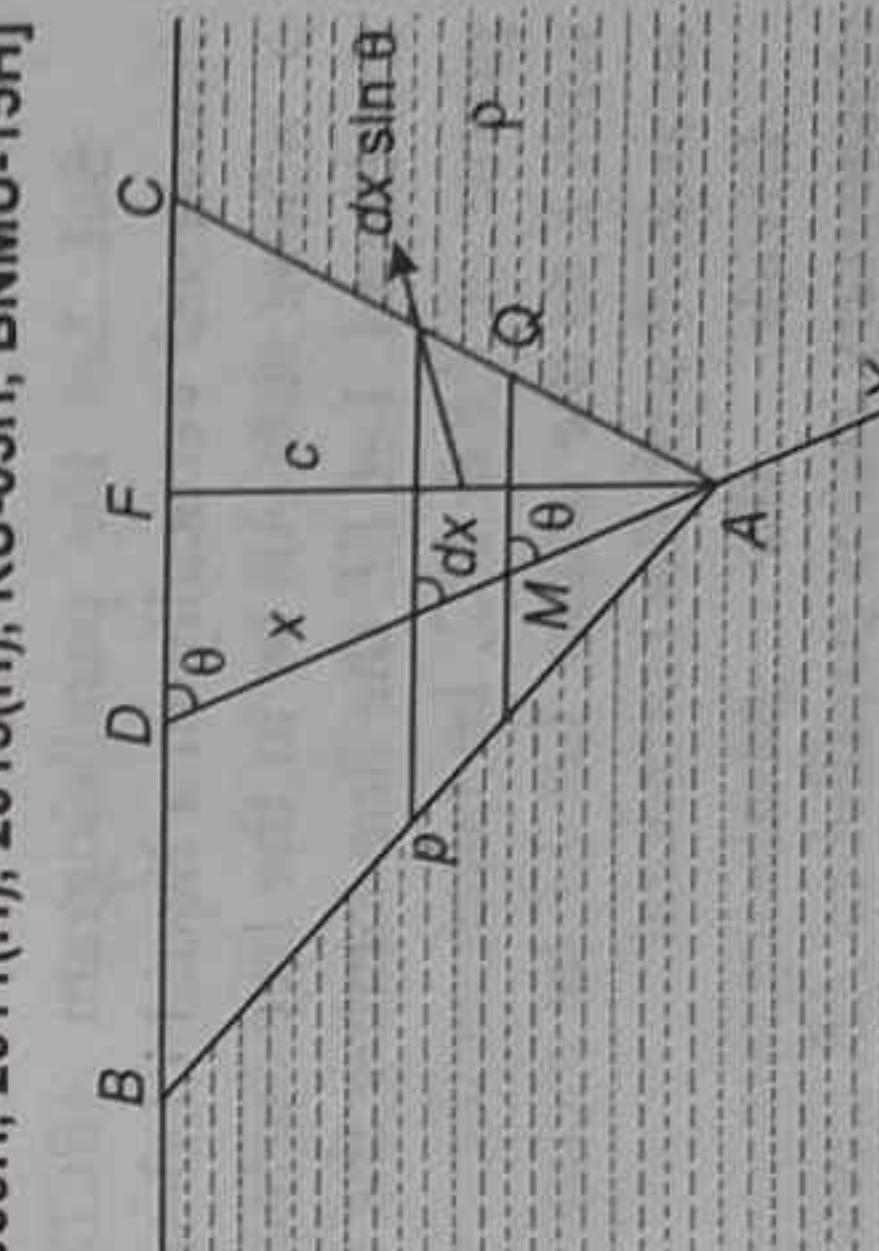


Fig. 72

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Divide the lamina into elementary strips parallel to the base  $BC$ . Consider an elementary strip  $PQ$  of width  $dx$  at a distance  $x$  from  $D$  measured along the median.

$$\begin{aligned} \frac{PQ}{BC} &= \frac{AM}{AD} = \frac{AD - DM}{AD} = \frac{r - x}{r} \\ PQ &= \frac{(r - x)}{r} BC = \frac{(r - x) a}{r} \end{aligned}$$

The thrust on elementary strip  $PQ$  is given by

$$\begin{aligned} dT &= PQ \cdot dx \sin\theta \rho g x \sin\theta \\ &= \left( \frac{r - x}{r} \right) \rho g x \sin^2 \theta dx \end{aligned}$$

Let  $\bar{x}$  be the distance of the C.P. from the origin  $D$ , then

$$\begin{aligned} \bar{x} &= \frac{\int x dT}{\int dT} = \frac{\int_0^r x \cdot \frac{(r - x)}{r} \rho g \sin^2 \theta \cdot x dx}{\int_0^r \frac{(r - x)}{r} \rho g \sin^2 \theta x dx} \\ &= \frac{\int_0^r (r - x) x^2 dx}{\int_0^r (r - x) x dx} \\ &= \frac{\left[ r \frac{x^3}{3} - \frac{x^4}{4} \right]_0^r}{\left[ r \frac{x^2}{2} - \frac{x^3}{3} \right]_0^r} = \frac{r}{2} = \frac{1}{2} AD \end{aligned}$$

$$\bar{x} = \frac{1}{2} AD \Rightarrow C.P. \text{ lies at the middle point of } AD.$$

$$\therefore \text{Depth of C.P. below the free surface} = \frac{1}{2} AD \sin\theta = \frac{1}{2} c.$$

**Example 6.** Find the C.P. of a parallelogram with one side in the free surface.

**[MU 2002H, B.N.M.U. 14H]**

Sol. Let the parallelogram  $ABCD$  be immersed vertically in a liquid of density  $\rho$ , whose side  $AB$  be in the free surface of the liquid. Let  $EF$  the line joining the mid points of  $AB$  &  $DC$ , be taken as the  $x$ -axis with  $E$  as the origin. Let  $EF$  be inclined at an angle  $\theta$  with the horizontal.

Divide the parallelogram into elementary strips parallel to  $AB$ . Consider an elementary strip  $PQ$  of width  $dx$  at a distance  $x$  from  $E$  measured along the slope of  $EF$ . Let  $EG = c$  be the depth of  $DC$  below  $AB$ .

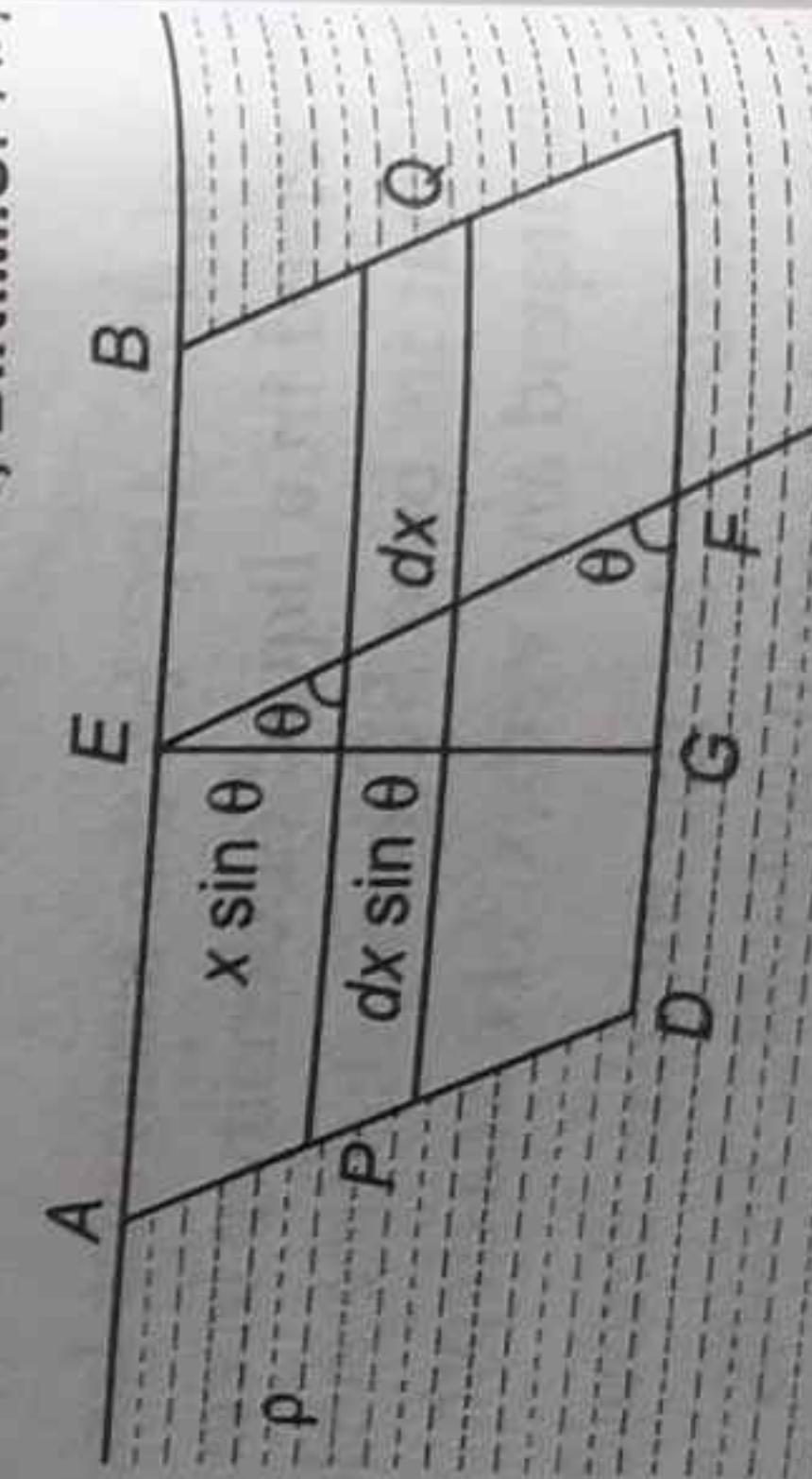


Fig. 73

Divide the parallelogram into elementary strips parallel to  $AB$ . Consider an elementary strip  $PQ$  of width  $dx$  at a distance  $x$  from  $E$  measured along the slope of  $EF$ . Let  $EG = c$  be the depth of  $DC$  below  $AB$ .

Since the strips are very thin, so the C.P. of each strip lies at their respective mid-points. Evidently C.P. of the whole lamina will be on the  $x$ -axis  $EF$ . If  $(\bar{x}, \bar{y})$  be the coordinates of C.P., then

$$\bar{x} = \frac{\int x \, dT}{\int dT} \quad \text{and} \quad \bar{y} = 0.$$

where  $dT = PQ \cdot dx \sin\theta \cdot x \sin\theta \rho g = PQ \cdot \rho g x \sin^2\theta \, dx$

$$\bar{x} = \frac{\int_0^{EF} x^2 \, dr}{\int_0^{EF} x \, dx} = \frac{\left[ \frac{x^3}{3} \right]_0^{EF}}{\left[ \frac{x^2}{2} \right]_0^{EF}} = \frac{2}{3} EF$$

$$\Rightarrow \bar{x} = \frac{2}{3} EF \text{ along } x\text{-axis}$$

$\therefore$  Depth C.P. below the surface  $= \bar{x} \sin\theta$

$$\begin{aligned} &= \frac{2}{3} EF \sin\theta \\ &= \frac{2}{3} EG. \end{aligned}$$

### § 4.5 : C.P of a combined figure

Let  $\bar{z}_1, \bar{z}_2, \bar{z}_3, \dots, \bar{z}_n$  be the depths of C.P. of different parts of lamina immersed in a liquid and  $T_1, T_2, T_3, \dots, T_n$  be the corresponding thrusts on the different parts. Let  $\bar{z}$  be the depth of C.P. of the combined lamina, then total thrust on the combined lamina is  $T_1 + T_2 + T_3 + \dots + T_n$  which acts normally at C.P.

By Varignon's theorem

$$\Rightarrow \bar{z}(T_1 + T_2 + \dots + T_n) = \bar{z}_1 T_1 + \bar{z}_2 T_2 + \dots + \bar{z}_n T_n$$

$$\Rightarrow \bar{z} \sum_{i=1}^n T_i = \sum_{i=1}^n \bar{z}_i T_i$$

$$\bar{z} = \frac{\sum_{i=1}^n \bar{z}_i T_i}{\sum_{i=1}^n T_i}$$

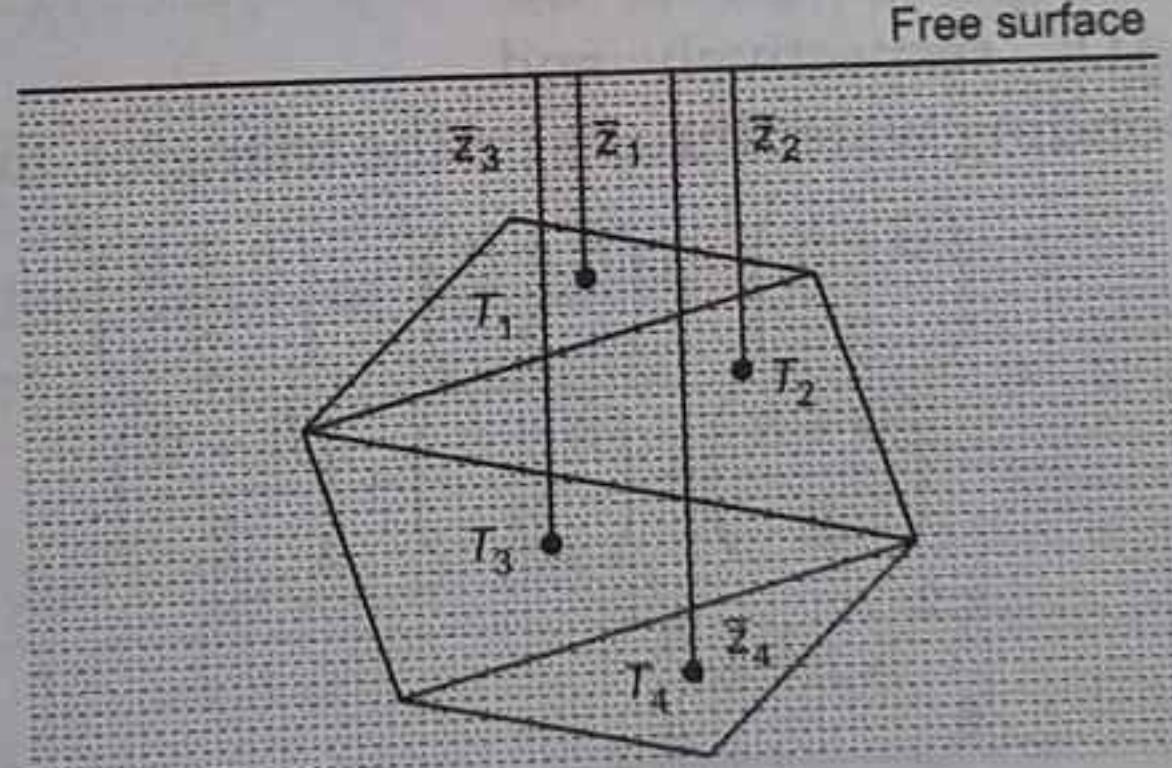


Fig. 74

This is the required formula for the depth of C.P. of the combined lamina.

### § 4.6 : C.P of Remaining Part of a lamina :

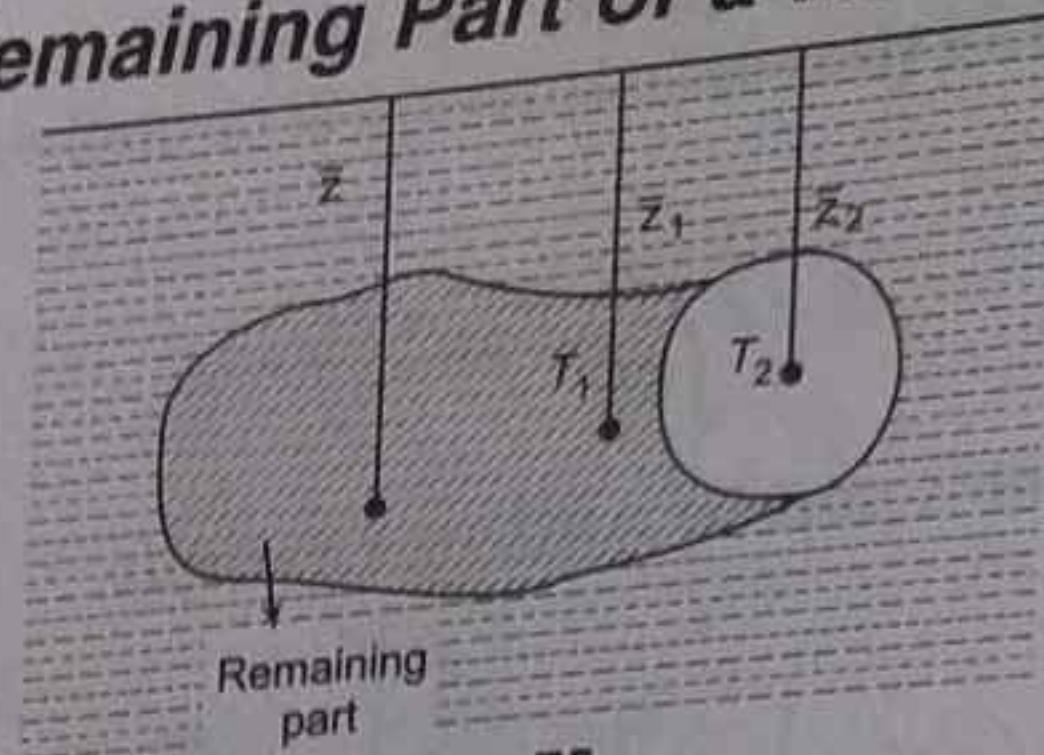


Fig. 75

Let  $\bar{z}_1, \bar{z}_2$ , be the depth of C.P of the whole lamina and cut off part of the lamina. Let  $T_1$  and  $T_2$  be the corresponding thrusts. Let  $\bar{z}$  be the depth of C.P of the remaining part of the lamina, then

$$\begin{aligned} z_1 &= \frac{\bar{z}(T_1 - T_2) + z_2 T_2}{(T_1 - T_2) + T_2} \\ z_1 &= \frac{\bar{z}(T_1 - T_2) + z_2 T_2}{T_1} \\ z_1 T_1 &= \bar{z}(T_1 - T_2) + z_2 T_2 \\ \bar{z} &= \frac{z_1 T_1 - z_2 T_2}{T_1 - T_2} \end{aligned}$$

This is the required C.P of remaining part of a Lamina.

## Worked Out Examples

\*Problems are of P.G. Standard.

**Example 7.** The lengths of two parallel sides of a trapezium are  $a$  &  $b$  and the distance between them is  $h$ . If the trapezium be immersed in water with its plane vertical and the side  $a$  in the surface. Prove that the C.P will be at a depth  $\frac{h(a+3b)}{2(a+2b)}$  below the surface.

[TMBU-2003(H)]

**Sol.** Let  $ABCD$  be the trapezium with height  $AL = h$ . Let the trapezium be immersed in a liquid in such a way that the side  $AD$  ( $= a$ ) be in the surface of the liquid. Joining the diagonal  $AC$  to divide the trapezium into two triangles.

Let  $z_1$  and  $z_2$  be the depth of C.P of the two triangles  $ABC$  &  $ADC$  respectively and  $T_1$  and  $T_2$  be the corresponding water thrusts on the triangles, then

$$\begin{aligned} \therefore z_1 &= \frac{\left(\frac{0+h}{2}\right)^2 + \left(\frac{h+h}{2}\right)^2 + \left(\frac{h+0}{2}\right)^2}{\left(\frac{0+h}{2}\right) + \left(\frac{h+h}{2}\right) + \left(\frac{h+0}{2}\right)} \\ &= \frac{\frac{h^2}{4} + h^2 + \frac{h^2}{4}}{\frac{h}{2} + h + \frac{h}{2}} \end{aligned}$$

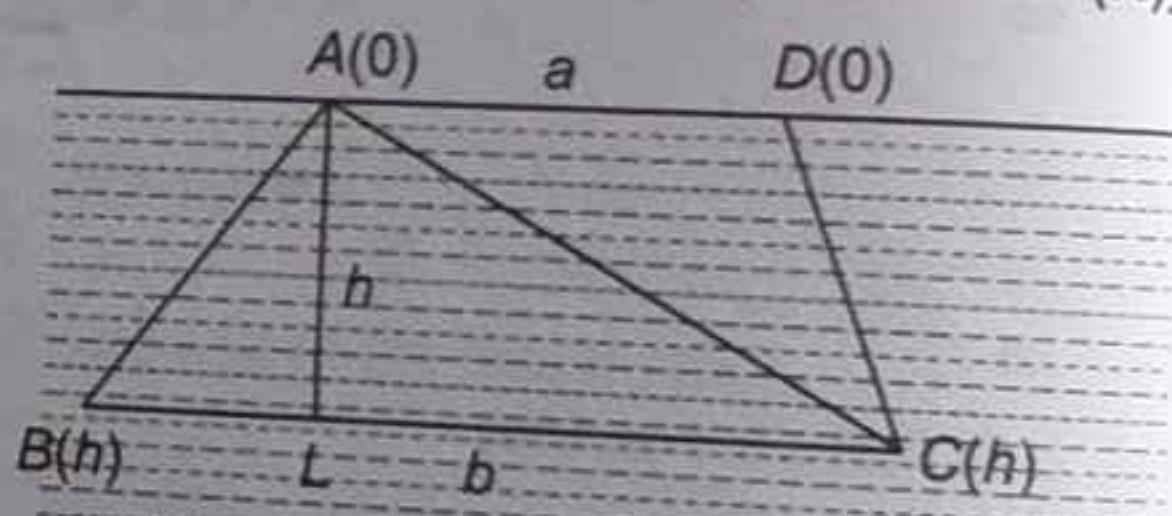


Fig. 76

$$= \frac{3}{4}h$$

Similarly  $\bar{z}_2 = \frac{h}{2}$

Now  $T_1 = \text{area} \times \text{pressure at C.G.}$

$$T_1 = \frac{1}{2}bh \times \rho g \left( \frac{0+h+h}{3} \right)$$

$$T_1 = \frac{1}{3}bh^2\rho g$$

$$T_2 = \frac{1}{2}ah \times \rho g \left( \frac{0+0+h}{3} \right)$$

$$= \frac{1}{6}a\rho gh^2$$

Let  $\bar{z}$  be the depth of C.P. of the trapezium, then

$$\bar{z} = \frac{z_1 T_1 + z_2 T_2}{T_1 + T_2}$$

$$\bar{z} = \frac{\frac{3}{4}h \times \frac{1}{3}h^2 b \rho g + \frac{h}{2} \times \frac{1}{6}a \rho g h^2}{\frac{1}{3}h^2 b \rho g + \frac{1}{6}a \rho g h^2}$$

$$= \frac{\frac{1}{4}h^3 \rho g \left( b + \frac{a}{3} \right)}{\frac{1}{3}h^2 \rho g \left( b + \frac{a}{2} \right)}$$

$$= \frac{3}{4}h \frac{2(3b+a)}{3(2b+a)}$$

$$\bar{z} = \frac{h}{2} \left( \frac{a+3b}{a+2b} \right) \quad \text{Proved.}$$

**Example 8.** A rhombus ABCD is completely immersed in a liquid with the vertex A in the surface and the diagonal AC through the vertex is vertical. Prove that the C.P. divides the diagonal AC in the ratio 7 : 5.

**Sol.** Let  $AG = GC = h$  then depth of B = depth of D =  $h$ , thus the depth of C.P. of the triangular ABC is given by

$$z_1 = \frac{0+h^2 + 2h^2 + 0.h + h.2h + 0.2h}{2(0+h+2h)}$$

$$z_1 = \frac{7h^2}{6h} = \frac{7h}{6}$$

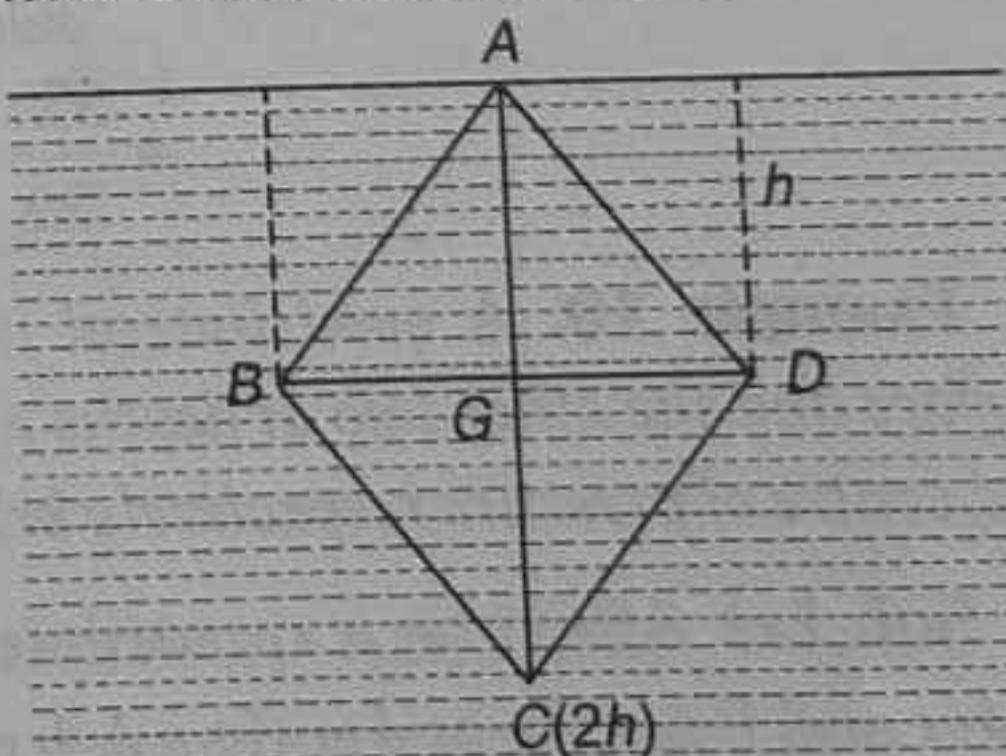


Fig. 77

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$$z_1 = \frac{7h}{6}$$

$z_1 = \frac{7h}{6}$  also given by  
 By symmetry of the depth of C.P of the triangle ACD is also given by

$$z_2 = \frac{7}{6}h$$

∴ The depth of C.P of the rhombus is given by  $\bar{z} = \frac{7}{6}h$  = depth or C.P of the

$\Delta ABC$  = depth of C.P of the  $\Delta ACD$  below the uppermost vertex A.

Height of C.P above the lowest vertex C  
 $= 2h - \frac{7}{6}h = \frac{5}{6}h$

∴ C.P divides the diagonal AC into ratio 7 : 5.

**Example 9.** A square is immersed with the diagonal vertical and its lowest point as deep again as its highest point. Find the depth of its C.P. [TMBU-2004H]

**Sol.** Let ABCD be a square lamina immersed in a liquid of density  $\rho$  in such a way that the depth of A below the free surface be  $h$ , then from the question, depth of C is  $2h$ .

∴ Depth of B = Depth of D =  $\frac{3}{2}h$

By symmetry, the C.P of the lamina lies on the diagonal AC.

∴ Depth of C.P of the square lamina ABCD = the depth of C.P. of the  $\Delta ABC$ .

$$\begin{aligned} & h^2 + \left(\frac{3}{2}h\right)^2 + (2h)^2 + h \cdot \frac{3}{2}h + \frac{3}{2}h \cdot 2h + h \cdot 2h \\ &= \frac{2\left(h + \frac{3}{2}h + 2h\right)}{2h} \\ &= \frac{h^2 + \frac{9h^2}{4} + 4h^2 + \frac{3}{2}h^2 + 3h^2 + 2h^2}{2h} \\ &= \frac{55h^2}{36h} \\ &= \frac{55h}{36} \end{aligned}$$

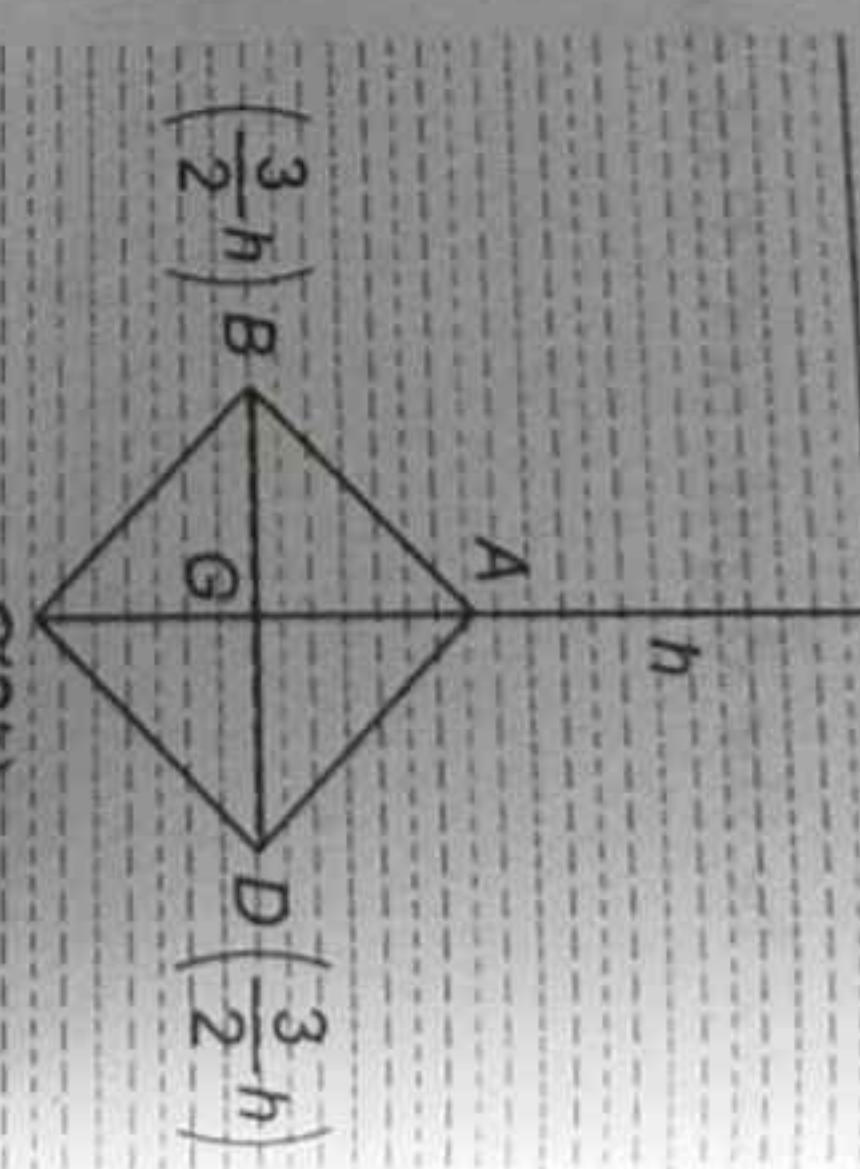


Fig. 78

**Example 10.** A quadrilateral is immersed in water with two angular points in the