

■ 4.7 : To find the C.P. of a lamina immersed in a liquid referred to coordinate axes lying in the plane of the lamina.

Sol. Let a plane lamina of area S be immersed in a liquid of density ρ , whose plane intersects the free surface in a straight line LM . Taking a point O on the line LM as the origin and $OLMY$ as the y -axis. A vertical line OX in the plane of the lamina is taken as the x -axis. Considering an elementary area ds at $P(x, y)$ with respect to the rectangular axes (O, XY) . Let dT be the elementary thrust on the elementary ds then

$$\begin{aligned} dT &= \text{area} \times \text{pressure at its C.G.} \\ &= \rho g x \, ds \end{aligned}$$

Let $C(\bar{x}, \bar{y})$ be the coordinates of C.P. of the lamina, at which the resultant thrust $\int dT$ will act normally.

Taking moment of all thrusts acting on the plane lamina about y -axis, we get

$$\begin{aligned} \bar{x} \int dT &= \int x \, dT && \text{(by Varignon's theorem)} \\ \bar{x} &= \frac{\int x \, dT}{\int dT} = \frac{\int x \rho g x \, ds}{\int \rho g x \, ds} = \frac{\int \rho x^2 \, ds}{\int \rho x \, ds} \end{aligned}$$

Similarly by taking moment of all thrusts and resultant thrust acting normally on the plane lamina, about x -axis.

we get,

$$\begin{aligned} \bar{y} \int dT &= \int y \, dT && \text{(by Varignon's theorem)} \\ \bar{y} &= \frac{\int y \rho g x \, ds}{\int \rho g x \, ds} = \frac{\int \rho xy \, ds}{\int \rho x \, ds} \end{aligned}$$

If the liquid is homogeneous then ρ is constant and, hence

$$\bar{x} = \frac{\int x^2 \, ds}{\int x \, ds}, \quad \bar{y} = \frac{\int xy \, ds}{\int x \, ds}.$$

If we take $ds = dx \, dy$, then

$$\bar{x} = \frac{\iint x^2 \, dx \, dy}{\iint x \, dx \, dy}, \quad \bar{y} = \frac{\iint xy \, dx \, dy}{\iint x \, dx \, dy},$$

Thus the coordinates of C.P. are $\left(\frac{\iint x^2 \, dx \, dy}{\iint x \, dx \, dy}, \frac{\iint xy \, dx \, dy}{\iint x \, dx \, dy} \right)$.

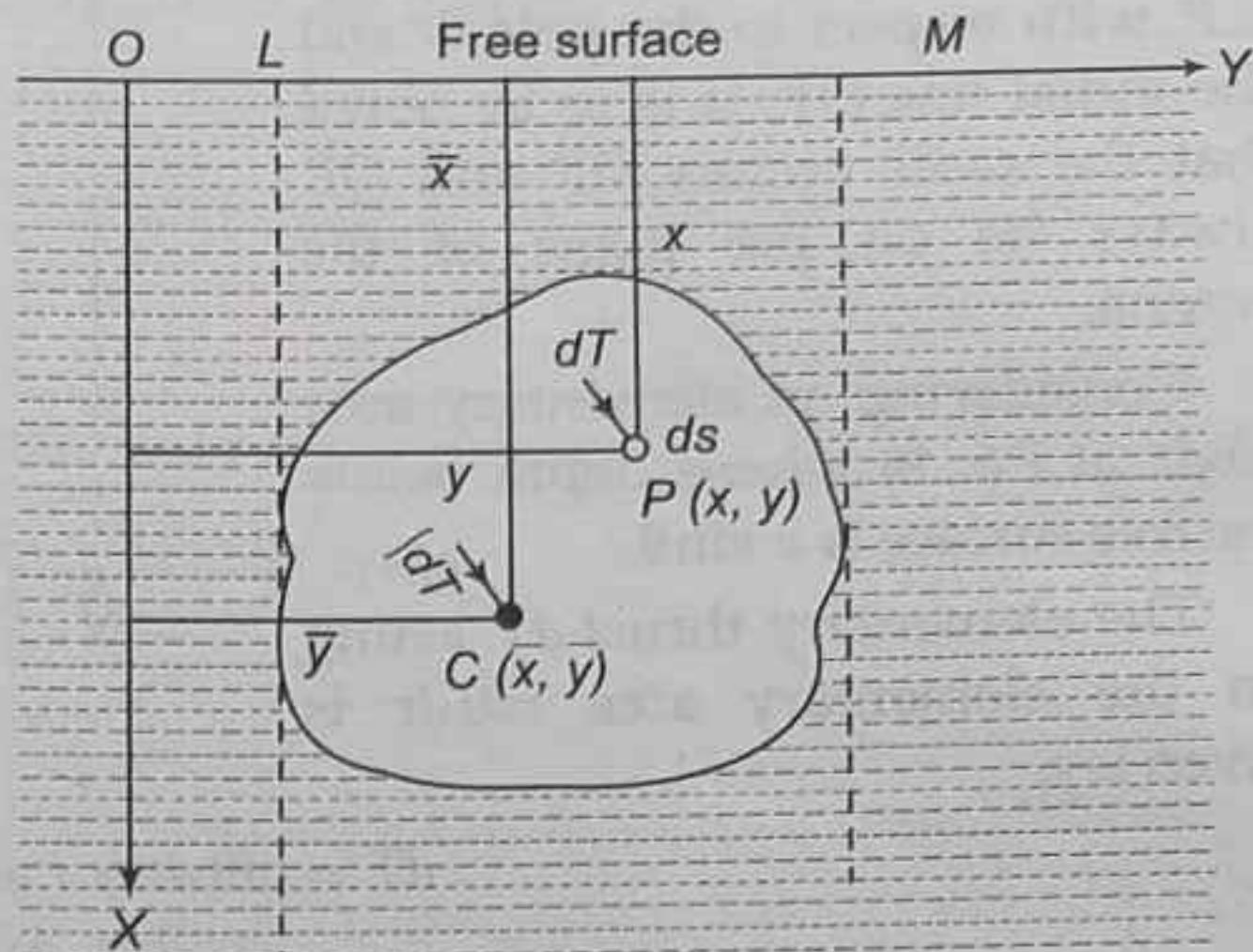


Fig. 88

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Polar coordinates of C.P. : Let (r, θ) be the polar coordinates of an arbitrary point P of the lamina and (R, ϕ) be the polar coordinates of C.P. with respect to the pole O and the initial line OY . It is to be noted that the radii vectors OP and OR always lie on the plane of the lamina.

Considering an elementary area $rd\theta dr$ at $P(r, \theta)$ whose depth below the free surface is $r \sin \theta$.

The elementary thrust dT acting on the elementary area $rd\theta dr$ is given by

$$dT = rd\theta dr \times r \sin \theta \rho g \\ = \rho g r^2 \sin \theta d\theta dr$$

Now taking moment of all the thrusts about the free surface, we get

$$R \sin \phi \int dT = \int r \sin \theta dT \quad (\text{by Varignon's theorem})$$

$$R \sin \phi = \frac{\iint r \sin \theta \rho g r^2 \sin \theta d\theta dr}{\iint \rho g r^2 \sin \theta d\theta dr} \\ = \frac{\iint \rho r^3 \sin^2 \theta d\theta dr}{\iint \rho r^2 \sin \theta d\theta dr} = \bar{x} \quad (\text{say})$$

Similarly by taking moment about the vertical line OX , we get

$$R \cos \phi \int dT = \int r \cos \theta dT \\ R \cos \phi = \frac{\iint r \cos \theta \rho g r^2 \sin \theta d\theta dr}{\iint \rho g r^2 \sin \theta d\theta dr} \\ = \frac{\iint \rho r^3 \sin \theta \cos \theta d\theta dr}{\iint \rho r^2 \sin \theta d\theta dr} = \bar{y} \quad (\text{say})$$

$$\text{Thus } R = \sqrt{\bar{x}^2 + \bar{y}^2}, \phi = \tan^{-1} \left(\frac{\bar{x}}{\bar{y}} \right).$$

The polar coordinates of C.P. are $\left(\sqrt{\bar{x}^2 + \bar{y}^2}, \tan^{-1} \left(\frac{\bar{x}}{\bar{y}} \right) \right)$.

If the liquid is homogeneous then ρ is constant, and

$$\left. \begin{aligned} R \sin \phi &= \frac{\iint r^3 \sin^2 \theta d\theta dr}{\iint r^2 \sin \theta d\theta dr} \\ R \cos \phi &= \frac{\iint r^3 \sin \theta \cos \theta d\theta dr}{\iint r^2 \sin \theta d\theta dr} \end{aligned} \right\}$$

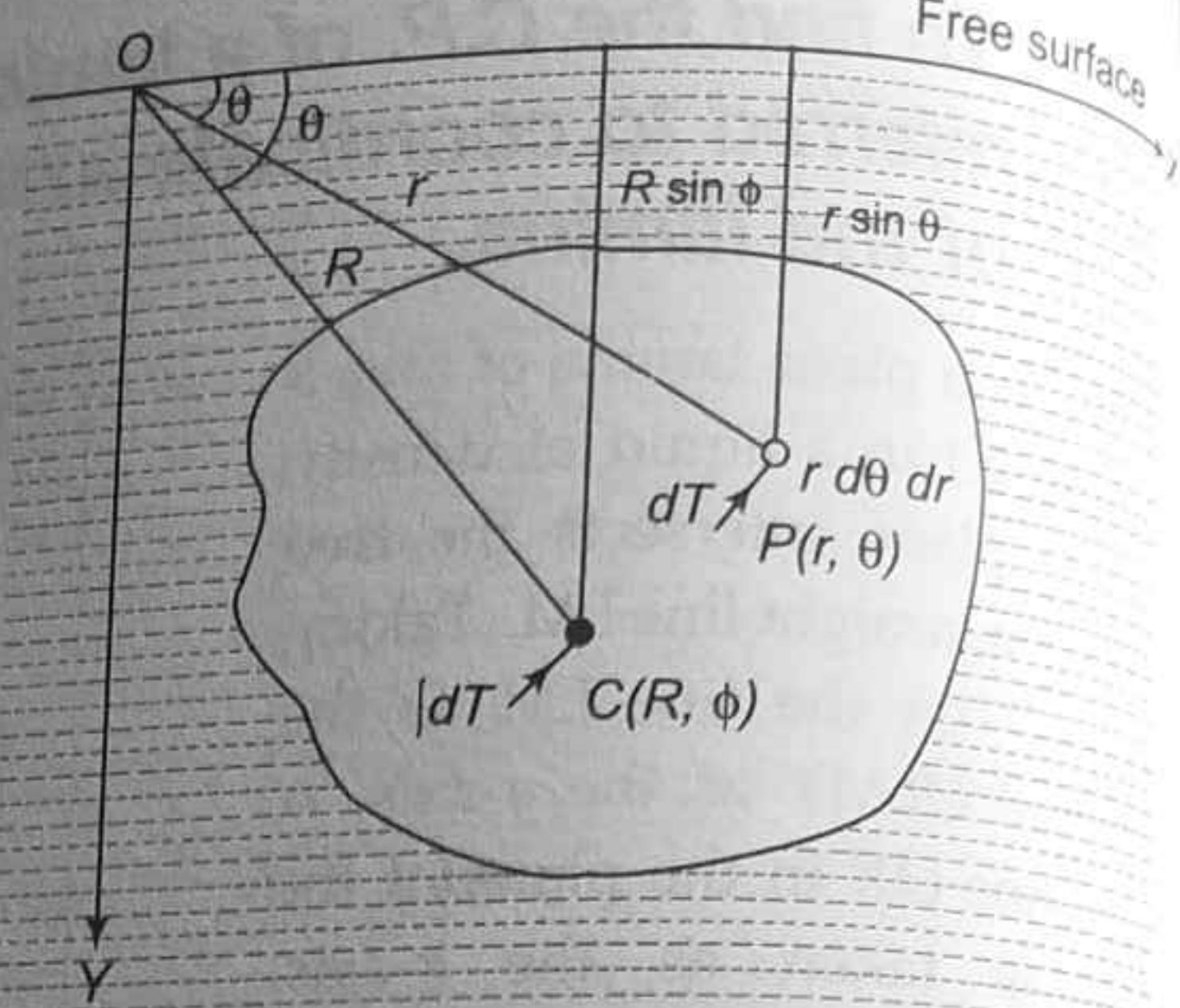


Fig. 89

Example 19. A semi circular lamina immersed in a liquid with a diameter into the free surface. Find the C.P of the lamina. [TMBU-2008(H), 2011(H), MU-02H]

Sol. Let a semi-circular lamina of radius a be immersed in a liquid of density ρ in such a way that the diameter AB of the lamina lies in the surface.

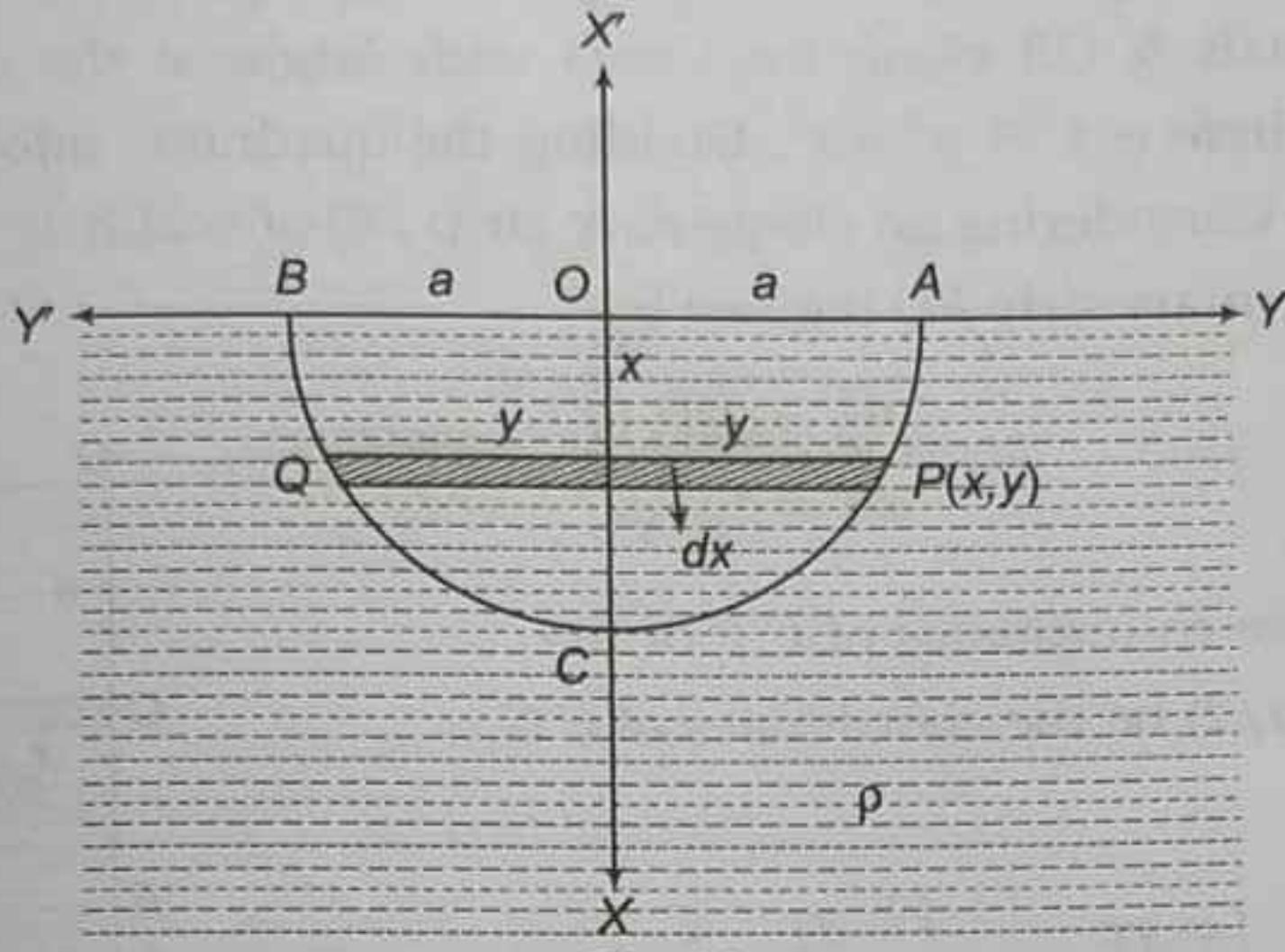


Fig. 90

Considering the central radius OCX as the x -axis and horizontal diameter AOB as the y -axis. The equation of the circle is $x^2 + y^2 = a^2$. Taking an elementary strips PQ parallel to diameter AOB at $P(x, y)$. If dx be the width of the strip then thrust on the elementary strip PQ is given by

$$dT = (2y dx) \rho g x$$

$$dT = 2g \rho x \sqrt{a^2 - x^2} dx$$

Let $(\bar{x}, 0)$ be the C.P then

$$\bar{x} = \frac{\int x dT}{\int dT}$$

$$= \frac{\int_0^a x^2 \sqrt{a^2 - x^2} dx}{\int_0^a x \sqrt{a^2 - x^2} dx}$$

Putting $x = a \sin \theta$, then $dx = a \cos \theta d\theta$, when $x = a$ then $\theta = \frac{\pi}{2}$, $x = 0$, then $\theta = 0$

$$\bar{x} = \frac{\int_0^{\pi/2} a^4 \sin^2 \theta \cdot \cos^2 \theta d\theta}{\int_0^{\pi/2} a^3 \sin \theta \cos^2 \theta d\theta}$$

$$= \frac{3\pi a}{16} \quad (\text{by reduction formula})$$

$\therefore \left(\frac{3\pi a}{16}, 0 \right)$ is the required C.P of the semi-circular lamina

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Example 20. Find the C.P of quadrant of a circle immersed vertically in a liquid of density ρ , when one radius being on the surface.

Sol. Let a quadrant OAB of circle of radius a be immersed vertically in a liquid such a way that the edge OB be in the surface of the liquid. Considering the line OA along the x -axis & OB along the y -axis with origin at the centre O , then equation of the circle is $x^2 + y^2 = a^2$. Dividing the quadrant into elementary strips parallel to y -axis. Considering an elementary strip PQ of width dx at $P(x, y)$ then thrust on the elementary strip PQ is given by

$$dT = y \cdot dx \cdot \rho g x$$

$$dT = \rho g x \sqrt{a^2 - x^2} dx$$

Let (\bar{x}, \bar{y}) be the co-ordinates of C.P of the quadrant and $(x, y/2)$ be the coordinates of C.G. of PQ, then

$$\bar{x} = \frac{\int x \cdot dT}{\int dT} \cdot \bar{y} = \frac{\int \frac{y}{2} \cdot dT}{\int dT}$$

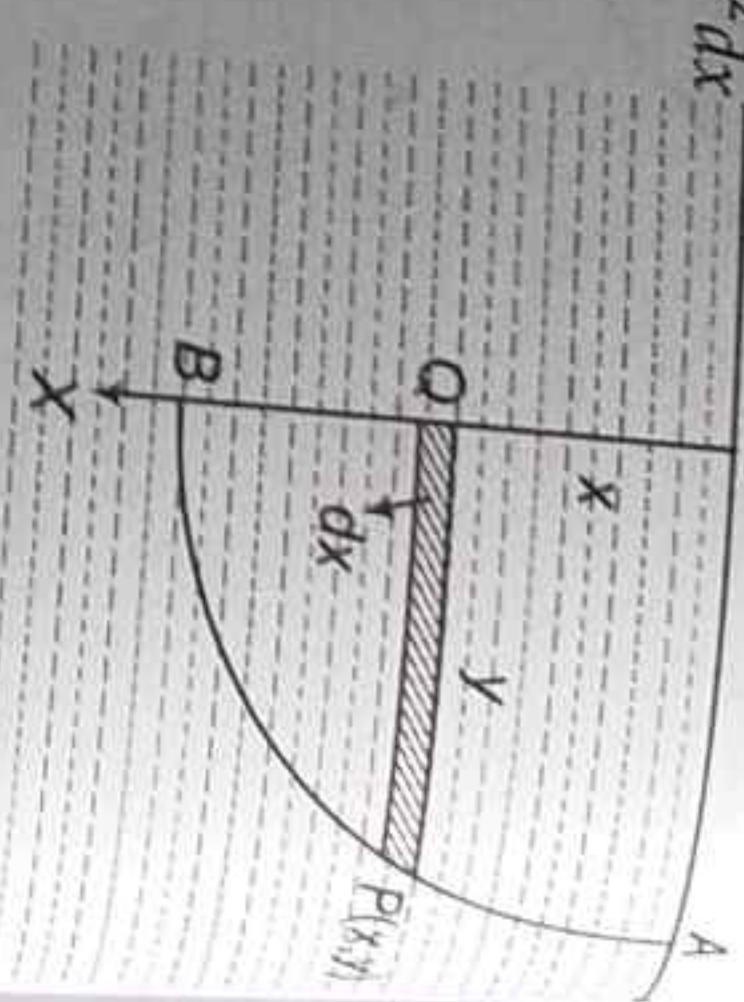


Fig. 91

$$\begin{aligned} \bar{x} &= \frac{\int x \cdot \rho g x \sqrt{a^2 - x^2} dx}{\int \rho g x \sqrt{a^2 - x^2} dx} \\ &= \frac{\int x^2 \sqrt{a^2 - x^2} dx}{\int x \sqrt{a^2 - x^2} dx} = \frac{3a\pi}{16} \end{aligned}$$

Also,

$$\begin{aligned} \bar{y} &= \frac{2 \int \rho g x \sqrt{a^2 - x^2} dx}{\int (a^2 - x^2) \cdot \rho g x \sqrt{a^2 - x^2} dx} \\ &= \frac{2 \int x \sqrt{a^2 - x^2} dx}{\int (a^2 - x^2) \cdot x dx} \\ &= \frac{\int (a^2 - x^2) \cdot x dx}{\int x \sqrt{a^2 - x^2} dx} \end{aligned}$$

element is given by

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TMBU-2007(H), 2007

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Putting $x = a \sin\theta$ then $dx = a \cos\theta d\theta$ if $x = a$, then $\theta = \frac{\pi}{2}$ and $x = 0$ then $\theta = 0$.

$$\bar{y} = \frac{\int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta) a \sin\theta \cdot a \cos\theta \cdot d\theta}{2 \int_0^{\pi/2} a \sin\theta \sqrt{a^2 - a^2 \sin^2 \theta}}$$

$$= \frac{\int_0^{\pi/2} \cos^3 \theta \cdot \sin\theta \cdot d\theta}{2 \int_0^{\pi/2} \sin\theta \cos^2 \theta \cdot d\theta} = \frac{a}{2} \cdot \frac{3}{4} = \frac{3a}{8} \quad (\text{by reduction formula})$$

∴ The co-ordinates of C.P $\equiv \left(\frac{3a\pi}{16}, \frac{3a}{8} \right)$.

Example 21. A quadrant of circle is just immersed vertically with one edge in the surface of a liquid whose density varies as depth. Find the co-ordinates of C.P. [MU-2000H, CU-2001(H)]

Sol. Let a quadrant OAB of circle of radius a be immersed vertically in a liquid of variable density, in such a way that the edge OB be in the surface of the liquid. Considering the radius along OA as the x-axis & OB as the y-axis with origin at the centre O, then the equation of the circle $x^2 + y^2 = a^2$.

Dividing the quadrant into elementary strip parallel to y-axis. Considering an elementary strip PQ of width dx at P(x, y) then the thrust on the elementary strip PQ is given by

$$dT = y \cdot dx \cdot \rho \cdot gx.$$

But from the question, density of the liquid varies as depth so $\rho \propto x$ i.e., $\rho = kx$

$$\therefore dT = ydx \cdot kxgx = kgx^2 \sqrt{a^2 - x^2} \cdot dx$$

Let (\bar{x}, \bar{y}) be the co-ordinates of C.P of the quadrant and $(x, y/2)$ be the co-ordination of C.G. of the strip PQ, then

$$\begin{aligned} \bar{x} &= \frac{\int x \cdot dT}{\int dT} \quad , \quad \bar{y} = \frac{\int \frac{y}{2} dT}{\int dT} \\ &= \frac{\int x \cdot kg x^2 \sqrt{a^2 - x^2} dx}{\int kg x^2 \sqrt{a^2 - x^2} dx} \\ &\therefore \bar{x} = \frac{0}{0} \frac{\int kg x^2 \sqrt{a^2 - x^2} dx}{\int kg x^2 \sqrt{a^2 - x^2} dx} \end{aligned}$$

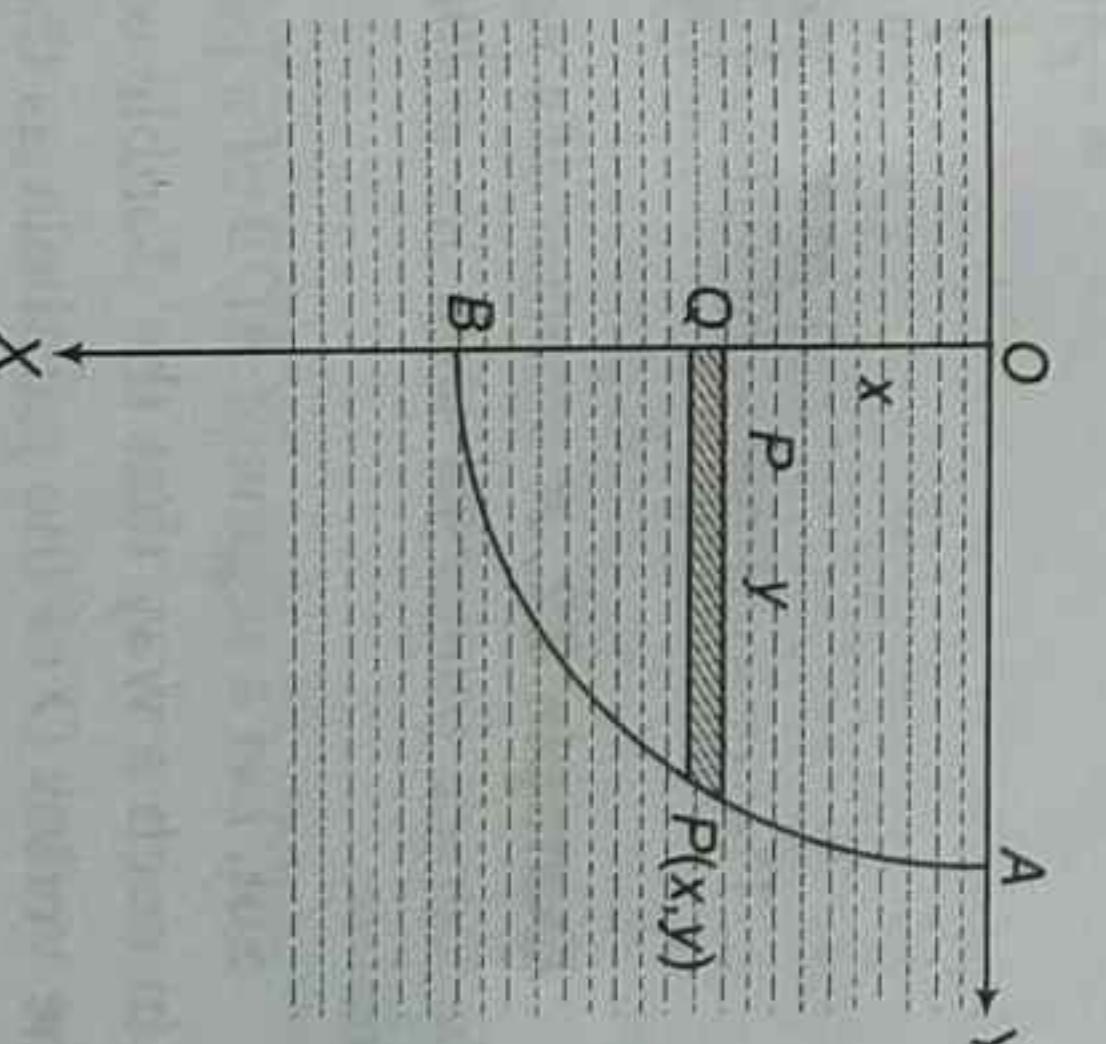


Fig. 92

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$$= \frac{\int_0^a x^3 \sqrt{a^2 - x^2} dx}{\int_0^a x^2 \sqrt{a^2 - x^2} dx}$$

Putting $x = a \sin \theta, dx = a \cos \theta d\theta$,
when $x = a$, then $\theta = 0, x = a$ then $\theta = \frac{\pi}{2}$

$$\bar{x} = \frac{\int_0^{\pi/2} a^3 \sin^3 \theta \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta}{\int_0^{\pi/2} a^2 \sin^2 \theta \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta}$$

$$= \frac{a \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta d\theta}{\int_0^{\pi/2} \sin^2 \theta \cdot \cos^2 \theta d\theta} = \frac{32a}{15\pi} \quad (\text{by using gamma function})$$

Again

$$\bar{y} = \frac{\int y \cdot dT}{\int dT}$$

$$= \frac{\int_0^a (a^2 - x^2) \cdot x^2 dx}{2 \int_0^a x^2 \sqrt{a^2 - x^2} dx}$$

$$= \frac{\int_0^{\pi/2} a^2 \sin^2 \theta \cdot (a^2 - a^2 \sin^2 \theta) a \cos \theta d\theta}{\int_0^{\pi/2} a^2 \sin^2 \theta \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta}$$

$$= \frac{a \int_0^{\pi/2} \sin^2 \theta \cdot \cos^3 \theta d\theta}{\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta} = \frac{16a}{15\pi} \quad (\text{by using gamma function})$$

\therefore The co-ordinates of C.P. = $\left(\frac{32a}{15\pi}, \frac{16a}{15\pi} \right)$.

Example 22. A segment of a parabola cut off by a double ordinate at a distance h from the vertex immersed with its ordinate in the surface of the liquid. Find the C.P. of the lamina.

Sol. Let a segment ADB of a parabola be immersed vertically in a liquid of density ρ in such a way that the double ordinate AMB be in the surface of the liquid. If a is the distance of the focus from the vertex O of the parabola as the origin, OMX as the x -axis and YOY' as the y -axis. Then the equation of parabola is $y^2 = 4ax$.

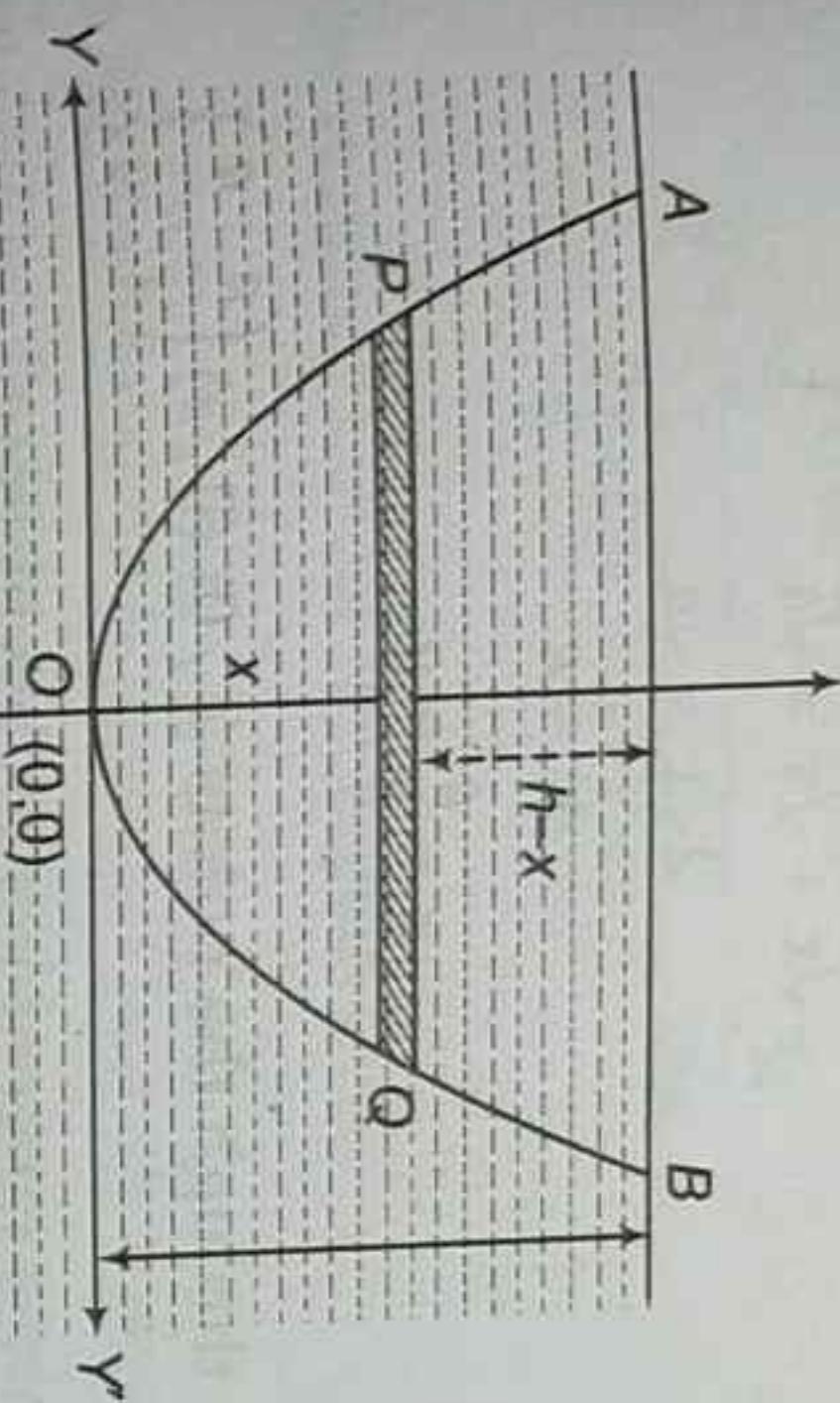


Fig. 93

Dividing the segment into elementary strips parallel to the y -axis.

Considering an elementary strip PQ of width dx at height x above the origin, then the thrust on the elementary strip is given by

$$dT = 2ydx (h - x) \rho g$$

$$= 2\rho g (h - x) \sqrt{4ax} dx$$

By symmetry, the C.P of the lamina lies on the x -axis, so $\bar{y} = 0$

$$\bar{x} = \frac{\int x dT}{\int dT}$$

$$= \frac{\int x \cdot 2\rho g (h - x) \sqrt{4ax} dx}{\int 2\rho g (h - x) \sqrt{4ax} dx}$$

$$= \frac{3h}{7}$$

Hence C.P of lamina is $\left(\frac{3h}{7}, 0\right)$.

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$$\Rightarrow \frac{\frac{\mu + y - \lambda - a}{1}}{\frac{1}{\lambda + a}} = \frac{\mu + z - \lambda - b}{\frac{1}{\lambda + b}} = \frac{(\mu - \lambda)}{\frac{1}{\lambda}}$$

$$\Rightarrow \frac{\frac{\mu + y - \lambda - a - \mu + \lambda}{1 - \frac{1}{\lambda}}}{\frac{1}{\lambda + a} - \frac{1}{\lambda}} = \frac{\mu + z - \lambda - b - \mu + \lambda}{\frac{1}{\lambda + b} - \frac{1}{\lambda}}$$

$$\Rightarrow \frac{(y - a)(\lambda + a)}{a} = \frac{(z - b)(\lambda + b)}{b}$$

$$\Rightarrow b(y - a)(\lambda + a) = a(z - b)(\lambda + b)$$

$$\Rightarrow b(y - a)\lambda - a(z - b)\lambda = a(z - b)b - b(y - a)a$$

$$\Rightarrow \lambda[by - ab - az + ab] = ab(z - b - y + a)$$

$$\Rightarrow \lambda = \frac{ab(z - b - y + a)}{by - az}$$

Example 25 A cistern of co-axial tanks.

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Example 28. Show that the centre of pressure of a circular area immersed in the liquid whose centre is at a depth h below the surface, when the density of the liquid varies as the depth, is at a depth $\frac{2a^2h}{a^2 + 4h^2}$ below the centre of the circle.

Sol. Let a be the radius and C be the centre of the circular area which is immersed in a liquid of variable density.

Taking C as the pole and the vertical radius CX as the initial line. Considering an elementary area $rd\theta dr$ at $P(r, \theta)$ then the elementary thrust dT on the elementary area is given by

$$dT = (h + r \cos\theta) \rho g rd\theta dr$$

$$\rho \propto h + r \cos\theta$$

$$\rho = \lambda (h + r \cos\theta)$$

$$dT = g\lambda (h + r \cos\theta)^2 rd\theta dr$$

But

i.e.,

∴

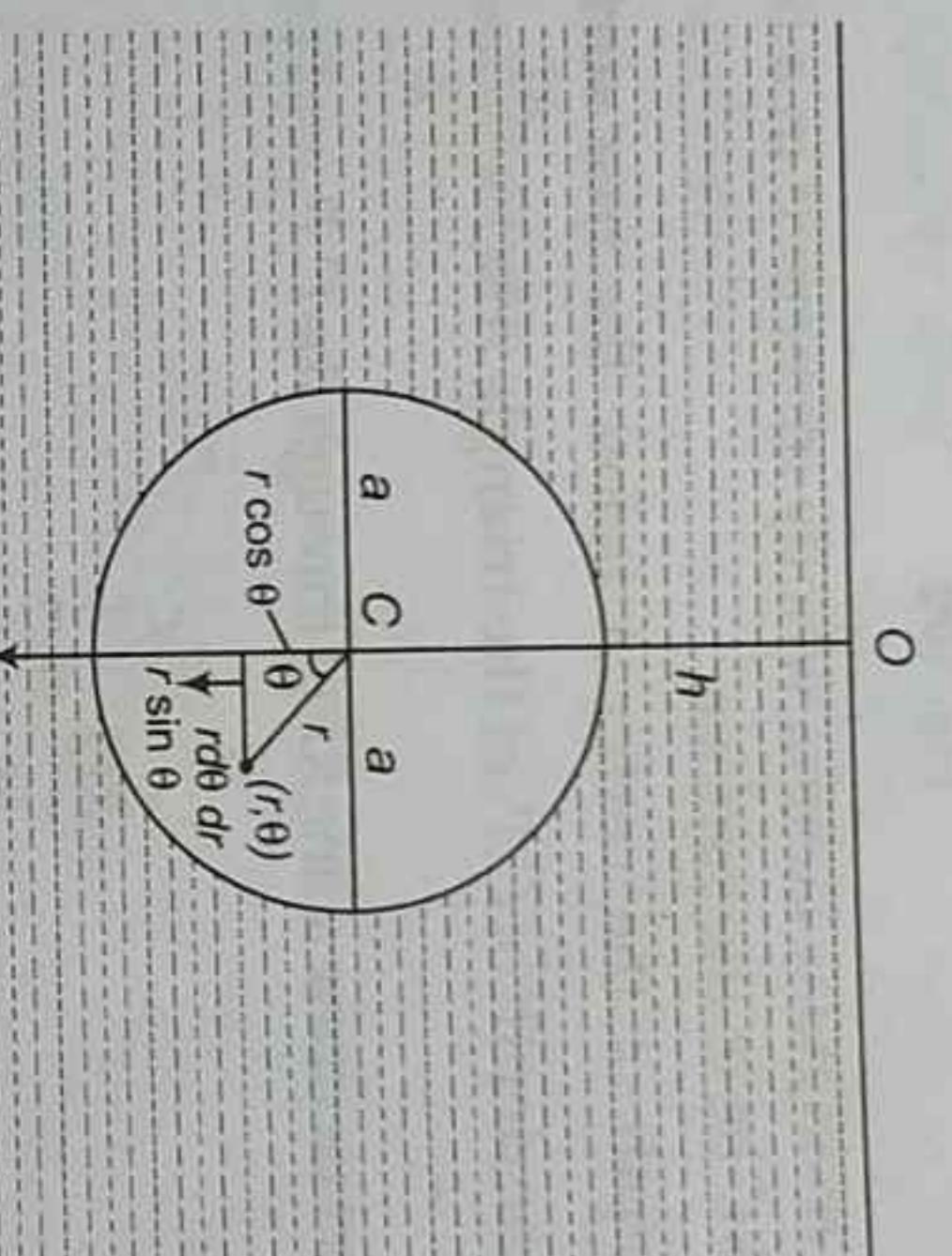


Fig. 99

∴ Depth of C.P. of the circular area below the centre

$$\begin{aligned}
 &= \frac{\iint r \cos\theta dT}{\iint dT} \\
 &= \frac{\iint_0^\pi r \cos\theta \cdot g\lambda (h + r \cos\theta)^2 r d\theta dr}{\iint_0^\pi g\lambda (h + r \cos\theta)^2 r d\theta dr} \\
 &= \frac{2 \int_0^{a\pi/2} \int_0^{2hr^3 \cos^2\theta} d\theta dr}{2 \int_0^{a\pi/2} \int_0^{h^2 + r^2 \cos^2\theta} r d\theta dr} \quad (\text{other integrals vanish as } \cos(\pi - \theta) = -\cos\theta) \\
 &= \frac{2h \left[\frac{r^4}{4} \right]_0^a \frac{1}{2} \frac{\pi}{2}}{h^2 \left[\frac{r^2}{2} \right]_0^a + \left[\frac{r^4}{4} \right]_0^a \frac{1}{2} \frac{\pi}{2}} = \frac{2a^2h}{(4h^2 + a^2)}
 \end{aligned}$$

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Example 29. A triangle is wholly immersed in a liquid with its base in the surface

[BNMU 13H]

Show that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two parts, the pressure on which are equal.

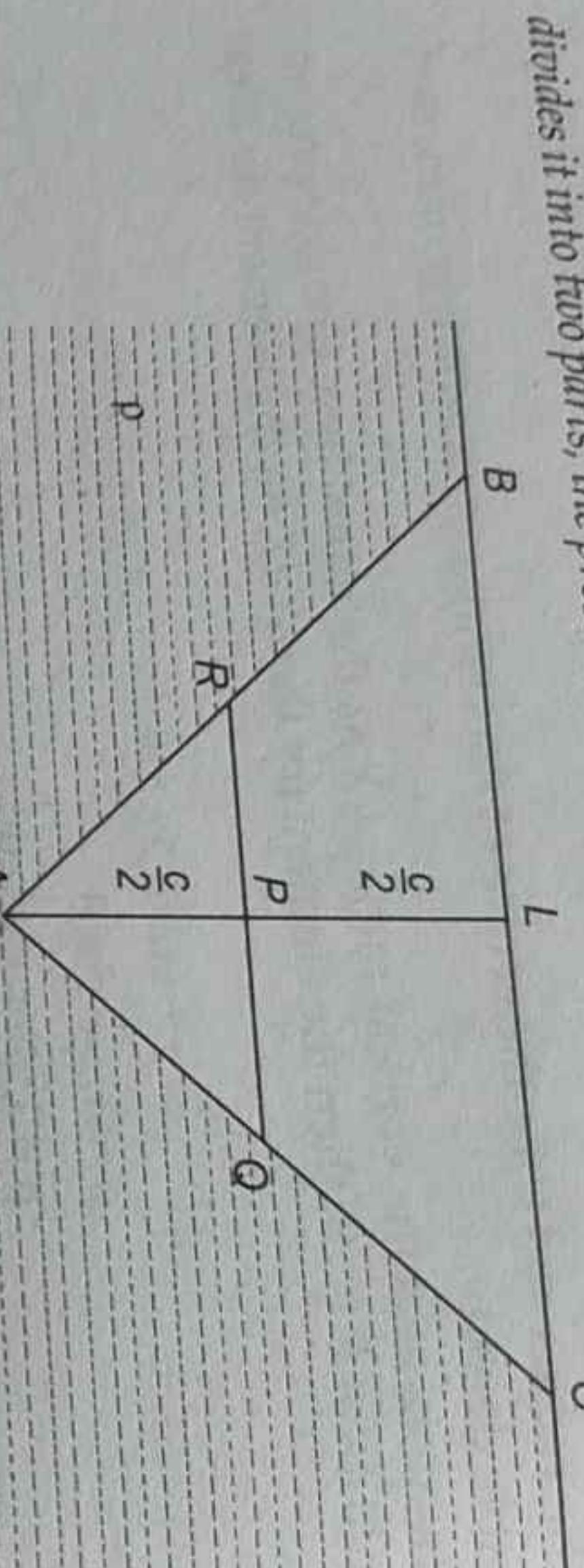


Fig. 100

Sol. Let ABC be the triangular lamina wholly immersed in a liquid of density ρ in such a way that its base BC be in the surface.

Let $AL = c$ be the depth of the vertex A below the free surface, then with the reference of example-3, the depth of C.P. of the triangle ABC is $\frac{AL}{2} = \frac{c}{2}$

Let $AP = PL = \frac{c}{2}$ and let us draw a line RPQ through the C.P. and parallel to BC , then

$$\begin{aligned} \frac{RQ}{BC} &= \frac{\frac{c}{2}}{c} \\ \Rightarrow \quad \frac{RQ}{BC} &= \frac{1}{2} \\ \Rightarrow \quad BC &= 2RQ. \end{aligned}$$

Depth of C.G. of the ΔABC is $\frac{c}{3}$

Depth of C.G. of the ΔAQR is $\frac{2c}{3}$

Thrust on the ΔABC is $\frac{1}{2} BC \cdot AL \times \rho g \left(\frac{c}{3}\right)$

$$= \frac{1}{6} BC \cdot \rho g c^2$$

Thrust on the ΔAQR is $\frac{1}{2} \cdot RQ \cdot AP \rho g \left(\frac{2c}{3}\right)$

$$= \frac{1}{12} BC \cdot \rho g c^2$$

\therefore Thrust on the quadrilateral $BCQRA$ is $\frac{1}{12} BC \rho g c^2$

\therefore Thrust on the quadrilateral $BCQRA$ = Thrust on ΔAQR .

Proved.

Example 30. If an area is bounded by two concentric semi circles with their common bounding diameter in the free surface, prove that the depth of the centre of pressure is $\frac{3}{16} \frac{\pi(a+b)(a^2+b^2)}{(a^2+b^2+ab)}$, where a and b are the radii.

$$\text{pressure is } \frac{3}{16} \frac{\pi(a+b)(a^2+b^2)}{(a^2+b^2+ab)}$$

Sol. Clearly the centre of pressure will lie on OY ; therefore $\bar{x} = 0$. Now consider an elementary area $r \delta\theta \delta r$. Depth of this element below the free surface $= r \sin\theta$

∴ Pressure on the elementary area

$$= \rho g r \sin\theta r \delta\theta \delta r.$$

$$\bar{y} = \frac{\int \int r \sin\theta \rho g r \sin\theta r d\theta dr}{\int \int \rho g r \sin\theta r d\theta dr}$$

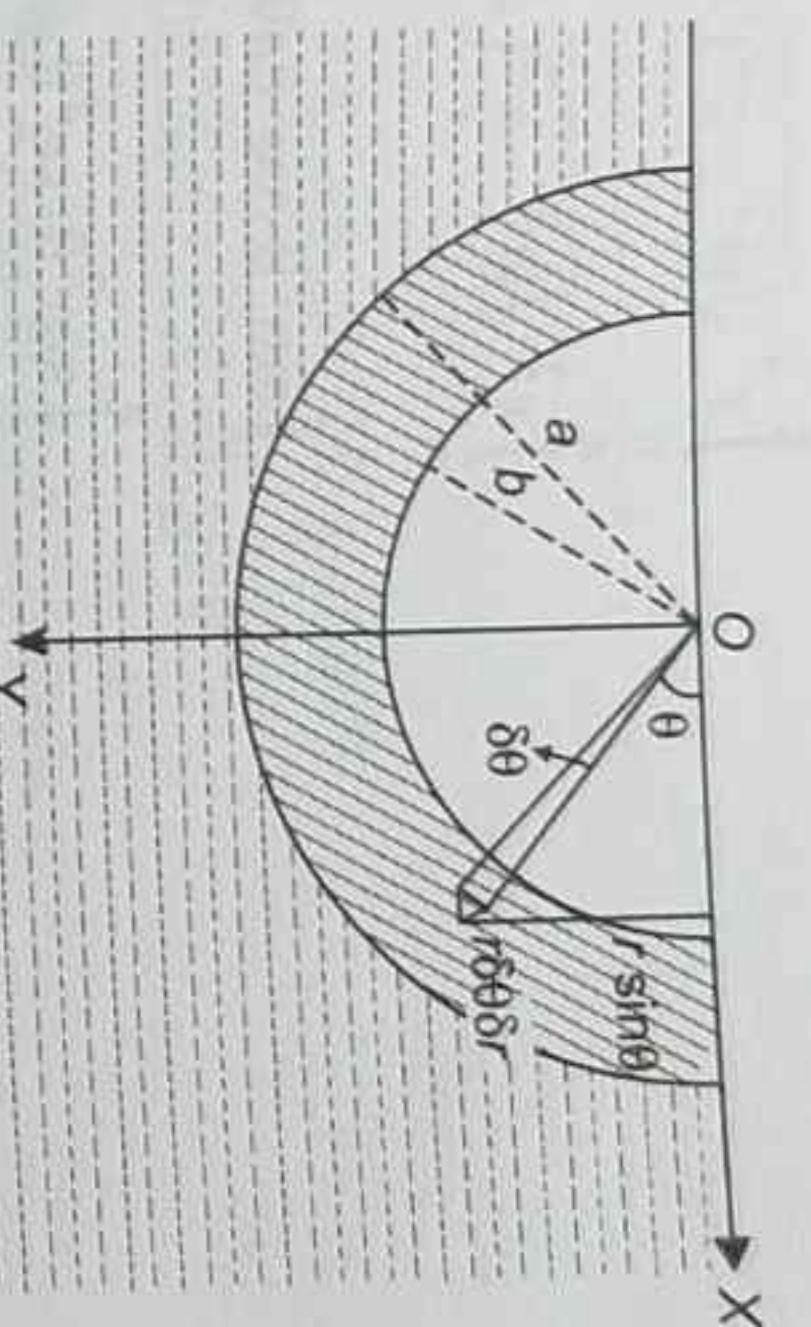


Fig. 101

$$= \frac{\int \int r^3 \sin^2\theta d\theta dr}{\int \int \rho^2 \sin\theta d\theta dr}$$

$$= \frac{\left[\frac{r^4}{4} \right]_b^a \cdot \frac{1}{2} \cdot \frac{\pi}{2}}{16(a^3 - b^3)} = \frac{3(a^4 - b^4)\pi}{16(a^2 + b^2 + ab)}$$

Example 31. A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth; if the minor axis be in the surface, find the eccentricity in order that the focus may be the centre of pressure. $\left[\text{An } e = \frac{32}{15\pi} \right]$ [TMBU-02H]

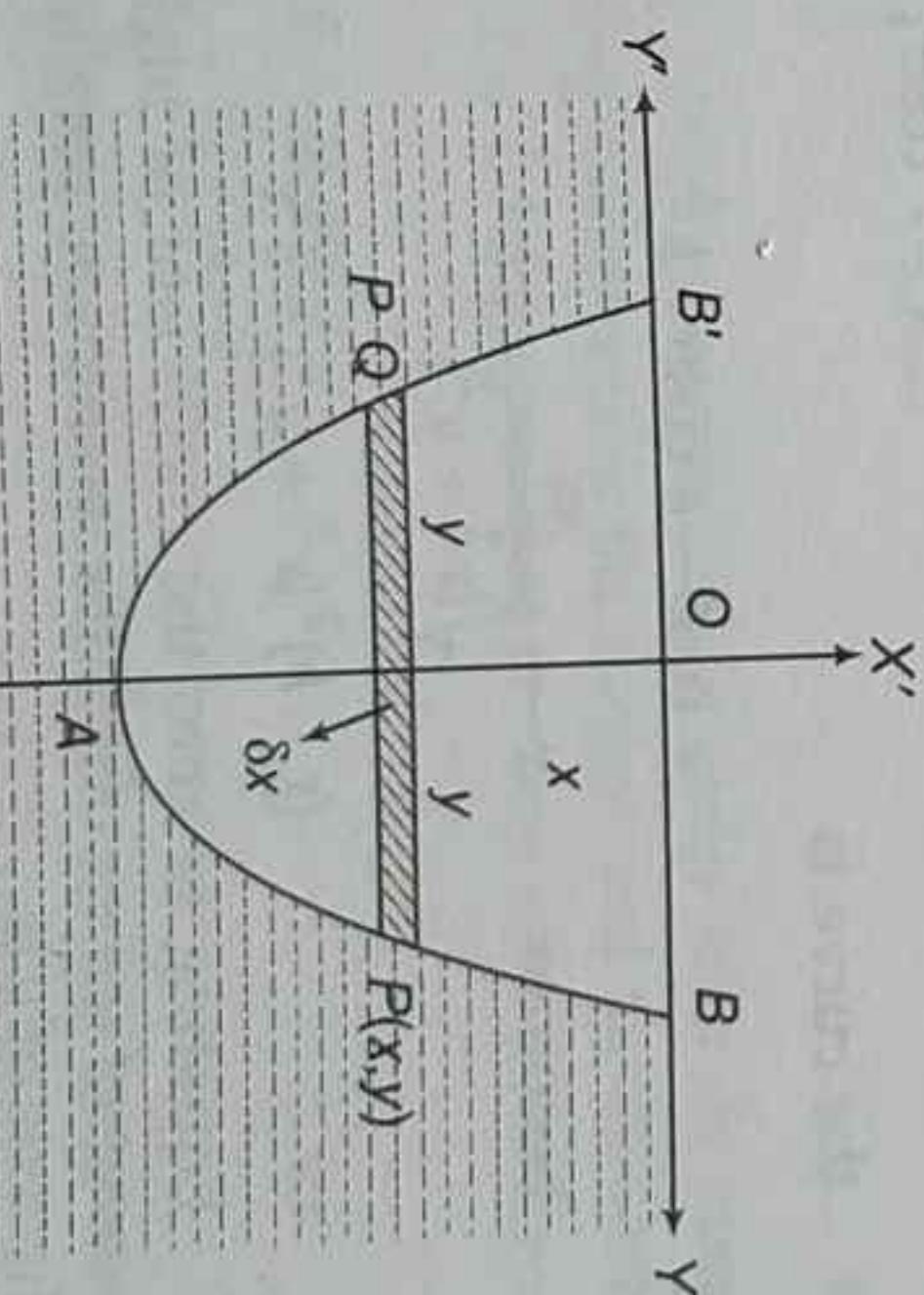
Sol. Referred to major and minor axes as axes of reference, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Now consider an elementary strip of breadth δx at a depth x below the free surface, then

$$\rho = \lambda x, (\text{area of strip}) = 2y\delta x.$$

Fig. 102



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\therefore pressure on the elementary strip

$$= \rho g x \cdot 2y \, dx = \lambda g x^2 \cdot 2y \, dx.$$

By symmetry, C.P. lies the axis of x . $\therefore \bar{y} = 0$

$$\text{and } \bar{x} = \frac{\int_a^b x \cdot \lambda g x^2 \cdot 2y \, dx}{\int_a^b \lambda g x^2 \cdot 2y \, dx}$$

$$\begin{aligned} &= \frac{\int_0^a x^3 \sqrt{(a^2 - x^2)} \, dx}{\int_0^a x^2 \sqrt{(a^2 - x^2)} \, dx}, \quad [\text{from (1) } y = \frac{b}{a}x \sqrt{(a^2 - x^2)}] \\ &= \frac{\int_0^{a/2} \sin^3 \theta \cos^2 \theta d\theta dx}{\int_0^{a/2} \sin^2 \theta \cos^2 \theta d\theta dx} \\ &= \frac{32a}{15\pi}. \end{aligned}$$

(by using gamma function)
(where $x = a \sin \theta$)

But C.P. coincides with the focus, so

$$ae = \frac{32a}{15\pi}$$

or

$$e = \frac{32}{15\pi} < 1 \quad \text{Ans.}$$

***Example 32.** Show the depth of the centre pressure of the area included between the arc and the asymptote of the curve $(r - a) \cos \theta = b$ is $\frac{a}{4} \cdot \frac{3\pi a + 16b}{3\pi b + 4a}$, the asymptote being in the surface and the plane of the curve vertical.

Sol. Changing to cartesian coordinates with the help of relations

$x = r \cos \theta, y = r \sin \theta,$
the curve is

$$r \cos \theta - a \cos \theta = b \quad \dots (1)$$

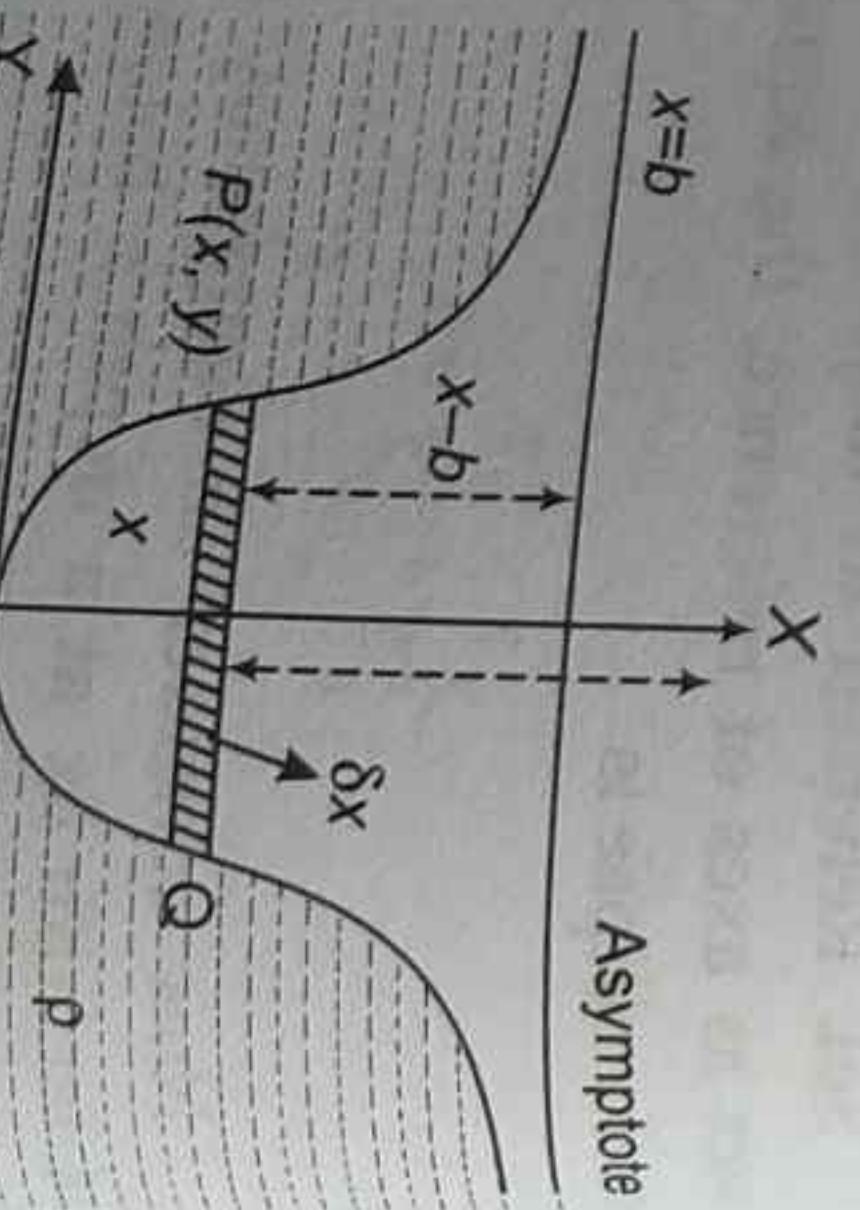
becomes

$$x - \frac{ax}{\sqrt{(x^2 + y^2)}} = b.$$

or

$$(x - b)^2 (x^2 + y^2) = a^2 x^2.$$

Equating to zero the coefficient of highest power of y , i.e., of y^2 , the coefficient of highest the axis of y is



Thus the line $x = b$ is in the surface.
 $x - b = 0$, i.e., $x = b$.

Fig. 103

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Now consider an elementary strip of breadth δx at a depth x above the origin. The area of the strip $= 2y \delta x$ and depth of the strip below free surface $= x - b$.

$$\therefore dT = p \, ds = \text{pressure on the strip} = \rho g (x - b) 2y \delta x.$$

By symmetry, $\bar{y} = 0$,

\bar{x} = depth of C.P. below the asymptote $x = b$

$$\begin{aligned} &= \frac{\int (x - b) p \, ds}{\int p \, ds} = \frac{\int (x - b) \rho g (x - b) 2y \, ds}{\int \rho g (x - b) 2y \, ds} \\ &= \frac{\int (x - b)^2 y \, dx}{\int (x - b) y \, dx}. \end{aligned}$$

Now (1) is

$$x - a \cos \theta = b \quad \text{or} \quad x - b = a \cos \theta, \quad dx = -a \sin \theta d\theta.$$

$$\therefore y = r \sin \theta = (a + b \sec \theta) \sin \theta \text{ as } r = a + b \sec \theta.$$

$$\begin{aligned} &= \frac{\int_0^{\pi/2} a^2 \cos^2 \theta (a + b \sec \theta) \sin \theta \cdot a \sin \theta \, d\theta}{\int_0^{\pi/2} a \cos \theta (a + b \sec \theta) \sin \theta \cdot a \sin \theta \, d\theta} \\ &= \frac{\int_0^{\pi/2} a \sin^2 \theta \cos^2 \theta \, d\theta + \int_0^{\pi/2} b \sin^2 \theta \cos \theta \, d\theta}{\int_0^{\pi/2} a \sin \theta \cos \theta \, d\theta + \int_0^{\pi/2} b \sin \theta \cdot a \sin \theta \, d\theta} \end{aligned}$$

$$\begin{aligned} &= \frac{\frac{\Gamma(3)\Gamma(\frac{3}{2})}{2\Gamma(3)} + b \frac{\Gamma(1)\Gamma(\frac{3}{2})}{2\Gamma(\frac{3}{2})}}{\frac{\Gamma(1)\Gamma(\frac{3}{2})}{2\Gamma(\frac{1}{2})} + b \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{2\Gamma(2)}} = \frac{a \frac{3\pi + 16b}{3b\pi + 4a}}{a \frac{\Gamma(\frac{5}{2})}{2\Gamma(\frac{3}{2})}} \\ &\quad \bullet \end{aligned}$$

Example 33. A semi circular lamina is completely immersed in water with its plane vertical, so that the extremity A of the bounding diameter is in the surface, and the diameter makes with the surface an angle α . Prove that if E be the centre of pressure and ϕ the angle between AE and the diameter.

$$\tan \phi = \frac{3\pi + 16 \tan \alpha}{16 + 15 \pi \tan \alpha}.$$

Sol. Let A be taken as origin and AX, the bounding diameter, as the axis of x . Hence the equation of the circle is

$$r = 2a \cos \theta.$$

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Consider an element $r \delta\theta \delta r$ about a point $P(r, \theta)$. Since AP makes an angle θ with AX and AX makes an angle α with free surface, the depth of the element below free surface

$$= r \sin(\theta + \alpha).$$

$\therefore dT = p ds$ = pressure on the element $= \rho g r \sin(\theta + \alpha) r \delta\theta \delta r$.

If (\bar{x}, \bar{y}) are the co-ordinates of the centre pressure E referred to (AX, AY) as axes then

$$\tan \phi = \frac{\bar{y}}{\bar{x}} = \frac{\int_0^{\pi/2} \int_0^{2a \cos \theta} r \sin \theta \rho g r \sin(\theta + \alpha) r d\theta dr}{\int_0^{\pi/2} \int_0^{2a \cos \theta} r \cos \theta \rho g r \sin(\theta + \alpha) r d\theta dr}$$

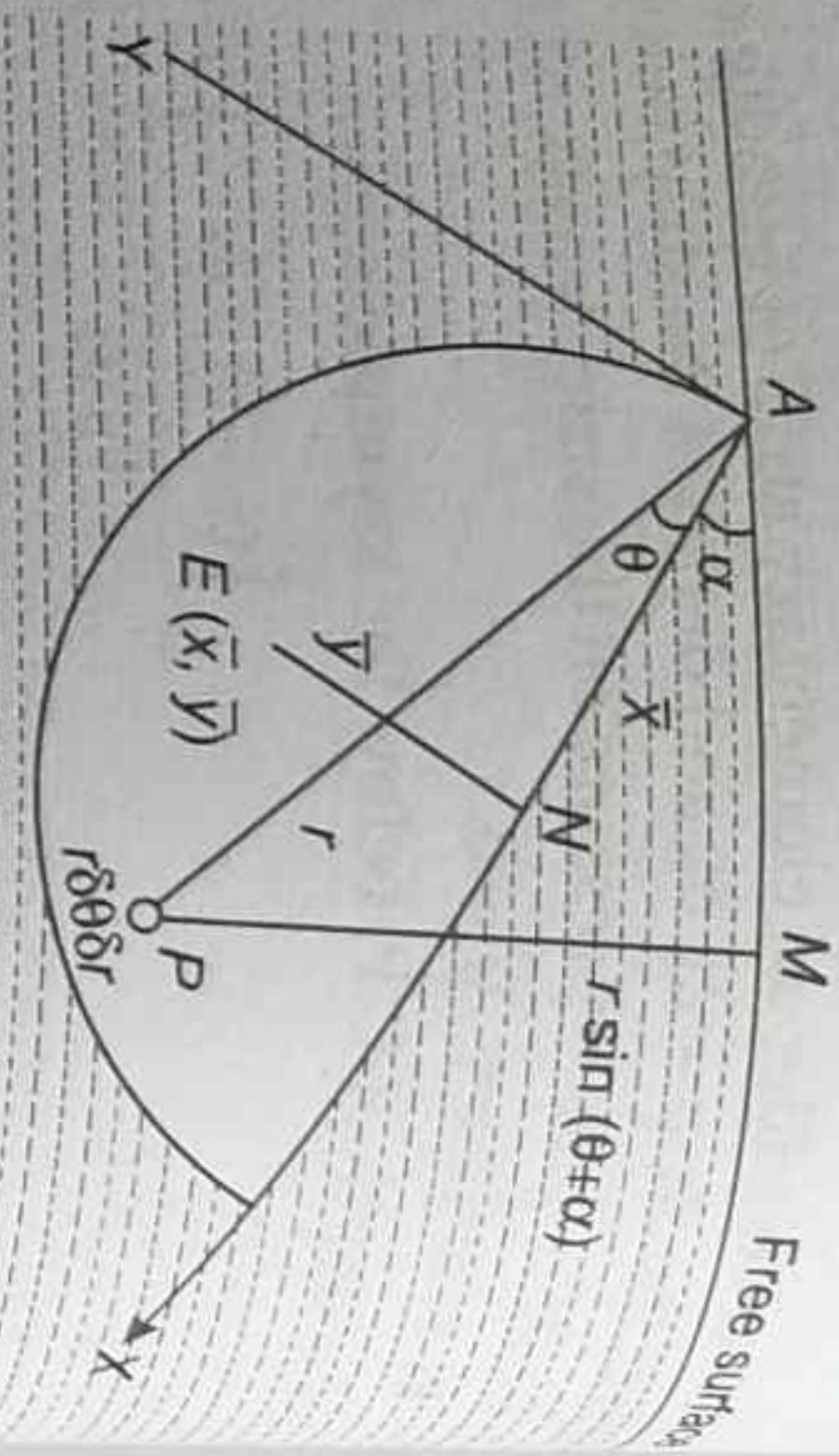


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Let (\bar{x}, \bar{y}) be the co-ordinates of the centre pressure E referred to (AX, AY) as axes then

$$\begin{aligned} \tan \phi &= \frac{\bar{y}}{\bar{x}} = \frac{\int_0^{\pi/2} \left[\frac{r^4}{4} \right]^{2a \cos \theta} \sin \theta \sin(\theta + \alpha) d\theta}{\int_0^{\pi/2} \left[\frac{r^4}{4} \right]^{2a \cos \theta} \cos \theta \sin(\theta + \alpha) d\theta} \\ &= \frac{\int_0^{\pi/2} \cos \theta (\sin \theta \sin(\theta + \alpha) + \cos \theta \sin(\theta + \alpha)) d\theta}{\int_0^{\pi/2} \cos^4 \theta (\sin \theta \cos \alpha + \cos \theta \sin \alpha) d\theta} \\ &= \frac{\int_0^{\pi/2} (\sin^2 \theta \cos^4 \theta \cos \alpha + \sin \theta \cos^5 \theta \sin \alpha) d\theta}{\int_0^{\pi/2} (\cos^5 \theta \sin \theta \cos \alpha + \cos^6 \theta \sin \alpha) d\theta} \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)} \cos \alpha + \left\{ -\frac{\cos^6 \theta}{6} \right\}_{0}^{\pi/2} \sin \alpha \\ &= \frac{\left[-\frac{\cos_6 \theta}{6} \right]_{0}^{\pi/2} \cos \alpha + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(4)} \sin \alpha}{16 \cos \alpha + 15 \sin \alpha} = \frac{3\pi + 16 \tan \alpha}{16 + 15 \tan \alpha}. \end{aligned}$$

Let have and i.e.

$$\begin{aligned} &\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)} \cos \alpha + \left\{ -\frac{\cos^6 \theta}{6} \right\}_{0}^{\pi/2} \sin \alpha \\ &= \frac{\left[-\frac{\cos_6 \theta}{6} \right]_{0}^{\pi/2} \cos \alpha + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(4)} \sin \alpha}{16 \cos \alpha + 15 \sin \alpha} = \frac{3\pi + 16 \tan \alpha}{16 + 15 \tan \alpha}. \end{aligned}$$