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PROJECT TITLE

Differential Equations and Its Aplications

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Examined by
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DECLARATION

- I , Binudhar Rabha hereby declare that project entitled "**Differential Eqautions and it's Applications"** submitted to Department Of Mathematics , Dudhnoi College,Dudhnoi for the degree of Bachelor of Science in Mathematics in faculty of science is a record of original work done by me under the supervision of Mr. Ripunjoy Choudhury ,Assistant professor, Department Of Mathematics ,Dudhnoi College. I would like to declare that neither the project nor any part thereof has been submitted to this college/institute or elsewhere for the award of any other degree or diploma.

Date-18-07-2022

- Place-Dudhnoi

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I also want to thank my family and classmates for their help and support carrying out the project.

Place-Dudhnoi

Date-18/07/2022

Binudhar Rabha

Signature

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CERTIFICATE

This is to certify that the project entitled "**Differential Equations and Its Applications**" is the outcome of the study and investigations carried out by **Binudhar Rabha,** It has been done under my supervision and guidance and neither the project nor any part thereof has been submitted to this or any other college for a Bachelor Of Science Degree.

Ripunjoy Choudhury 18/7/22

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ABSTRACT

This project work is mainly concern on the mathematical differential equation and its application. In this project, we want to highlight some mathematical problems in which process of differentiation are used. This project is written simply to illustrate on differentiation and their applications. The formation and classification of differentiation, the basic techniques of differentiation, list of derivatives and the basic applications of differentiations, which include motion, economic and chemistry.

Introduction to Differential Equations

DEFINITION 1.1 A differential equation (DE) is an equation in which an unknown function y(t) appears together with some of its derivatives.

In general, a DE can be written as

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0, \quad t \in I.$$

EXAMPLES.

(a) $y''(t) - 2y'(t) + y(t) - t^2 = 0, \qquad t \in (-1,1)$

(b) $y^{(4)}(t).y'(t) - y(t) = 2t + 1, \qquad t \geq 0.$

(c) $\frac{e^t y'(t)}{1+y^2(t)} = 5, \qquad t \in R.$

(d) Calculating the indefinite integral $\int 2t\, dt$ is the same as solving the differential equations $y'(t) = 2t$. Both problems ask for those functions, which have derivatives equal to 2t.

DEFINITION The order of a DE is defined by the highest derivative present in the equation.

Examples.

(a) The DE $y''(t) - (y'(t))^3 + 5y^6(t) = e^t$ has order 2.

(b) The DE $y^{(4)}(t) - y'(t) = 0$ has order 4.

Normal form of a DE. If the DE can be solved in the highest order derivative, then we say that we have obtained its normal form, which can be written as:

$$y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)), \quad t \in I.$$

Examples.

(a) The DE

$$t^2 y''(t) - t y'(t) + y(t) = e^t, \quad t \in [1, 2]$$

can be written in the following normal form:

$$y''(t) = \frac{1}{t} y'(t) - \frac{1}{t^2} y(t) + \frac{1}{t^2} e^t, \quad t \in [1, 2].$$

This normal form was obtained by dividing the DE by t^2. However, if we consider the inter- val $[-1, 1]$, dividing by t^2, which becomes 0 for $t = 0$, makes the normal form not defined on the entire interval $[-1, 1]$.

(b) The DE

$$e^{y'(t)} + y'(t) = (t + 1)y(t), \quad t \in [0, 1]$$

cannot be solved in $y'(t)$, so it cannot be written in normal form.

DEFINITION 2.1.3. A system of differential equations (SDEs) is formed by a number of differential equations involving more than one unknown functions and their derivatives.

Example of a SDEs:

$$\begin{cases} y'(t) = y(t) + z(t) \\ z'(t) = y(t) - z(t), \quad t \in R. \end{cases}$$

Note. Every higher order DE can be rewritten as a first order SDEs. This is very important for studying the existence of solutions and their numerical approximations.

Example.
Consider the second order DE $y''(t) = y(t)$ and introduce the function $z(t) = y'(t)$. Now we can write the SDEs

$$\begin{cases} y'(t) = z(t) \\ z'(t) = y(t), \end{cases}$$

which has a pair of solutions $(y(t), z(t))$, in which the first component is the same as the solution of the original second order DE and the second component is the derivative of it. Solving the SDEs is equivalent to solving the DE.

DEFINITION A solution of a DE on an interval **I** is a function $y = y(t)$ which, when substituted into the DE, satisfies the equation identically on the interval **I**.

Examples of solutions.

(a) $y(t) = \cos t$ is a solution of $y''(t) + y(t) = 0$ on $(-\infty, +\infty)$. To verify this we have to observe that $y''(t) = -\cos t$, and hence we get

$$-\cos t + \cos t = 0, \text{ for each } t \in (-\infty, +\infty),$$

which means that the $y(t) = \cos t$ satisfies the DE identically on (, +). But, observe also that it is not the only solution. $y_2(t) = \sin t$ is another solution. Moreover, any function of the form $y(t) = a \cos t + b \sin t$ is a solution.

(b) $y(t) = \sqrt{1 - t^2}$ is a solution of the DE $y'(t) \cdot y(t) + t = 0$ on the interval $(-1, 1)$, but it is not a solution on any interval larger than $(-1, 1)$.

Explicit and implicit solutions. Functions can be defined explicitly or implicitly. There- fore, solutions of DEs, which are functions,

hence, we can talk about explicit or implicit solutions. The above examples are all explicit solutions.
For an example of an implicit solution consider the equation

$$t^2 + y(t) + y^3(t) = 5 ,$$

which defines the function y(t) implicitly. If we use implicit differentiation, we get the DE $2t + y'(t) + 3y^2(t)\,y'(t) = 0$,

which has the same function y(t), as an implicitly defined solution.

Indefinite integrals: When we calculate the indefinite integral $\int 2t\,dt$, we actually solve the DE $y'(t) = 2t$. All the solutions are in the form $t^2 + c$, where the parameter c can be any real number. We can write this as $y(t) = t^2 + c$, and the meaning is that we have a one-parameter family of solutions, which is the same as the family of all the antiderivatives of 2t.

In general, DEs tend to have infinitely many solutions, but the general situation is much more complex.

Families of solutions:
If the solutions of a DE depend on parameters $c_1, \ldots, c_k$, then we call them a k-parameter family of solutions.

Singular solutions of DE.
A solution of a DE, which is not part of any family of solutions is called singular solution.

Examples of solutions for DEs.

(a) $y'(t) - y(t) = 0$ has solutions of the form $y(t) = ce^t$. Therefore, we have a one-parameter family of solutions and, as we will see later, all solutions are part of this family.

(b) $y''(t) - y(t) = 0$ has a two-parameter family of solutions of the form $y(t) = c_1e^t + c_2e^{-t}$.

(c) $y'(t) = t\sqrt{y(t)}$ has a one-parameter family of solutions $y(t) = \left(\frac{1}{4}t^2 + c\right)^2$, but also a solution $y(t) = 0$, which is not part of this family.

(d) $(y'(t))^2 + (y(t))^2 = 0$ has exactly one solution, the constant function $y(t) = 0$.

(e) $(y'(t))^2 + (y(t))^2 = -1$ does not have any solutions.

Solution curve of a DE.

The graph of a solution of a DE is called a solution curve. For example, $y_1(t) = e^t$, $y_2(t) = 0.5e^t$ and $y_3(t) = -0.4e^t$ are solutions of $y'(t) - y(t) = 0$, so their graphs, which are the curves with equations $y = e^t$, $y = 0.5e^t$ and $y = -0.4e^t$ are solution curves.

Initial value problems

Consider an n^{th}-order DE, $F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0$, $t \in I$, and fix $t_0 \in I$.

A system of initial conditions is a system of the form

$$y(t_0) = \alpha_0, \; y'(t_0) = \alpha_1, \ldots, y^{(n-1)}(t_0) = \alpha_{n-1},$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are n given numbers.

Initial Value Problems (IVP). The problem which combines a DE and a system of initial conditions is called an Initial Value Problem:

$$(IVP) \begin{cases} F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0, \quad t \in I \\ y(t_0) = \alpha_0 \\ y'(t_0) = \alpha_1 \\ \cdots\cdots\cdots \\ y^{(n-1)}(t_0) = \alpha_{n-1} \end{cases}$$

General solution of a DE: A n-parameter family of solutions of a n^{th}-order DE is called a general solution if for every system of initial conditions a member of that family solves the corresponding IVP.

Example. Consider the Initial Value Problem:

$$(IVP) \begin{cases} y''(t) - y(t) = 0, \quad -\infty < t < \infty \\ y(0) = 1 \\ y'(0) = 2. \end{cases}$$

The initial condition $y(0) = 1$ tells that the solution must go through the point (0, 1), while the condition $y'(0) = 2$ indicates that the slope of the tangent line to the solution curve at (0,1) must be 2.

The 2-parameter family of solutions

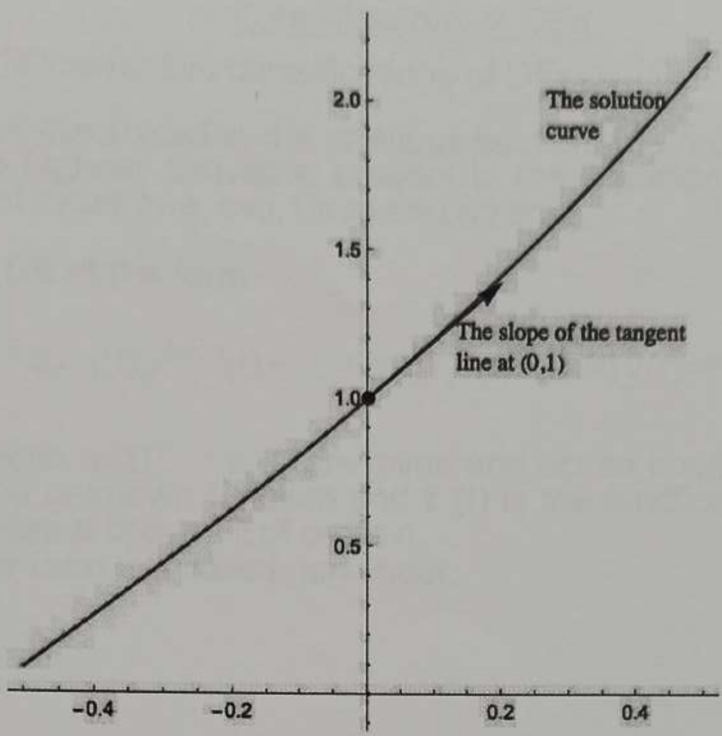
$$y(t) = ce^{t} + de^{-t},$$

is a general solution of the DE. The initial conditions lead to the linear system of equations

$$c + d = 1$$
$$c - d = 2.$$

Solving this system of linear equations gives $c = 3/2$ and $d = 1/2$. Therefore, this IVP has a unique solution of the form

$$y(t) = \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}.$$



2.0
1.5
1.0
0.5
-0.4
-0.2
0.2
0.4
The solution curve
The slope of the tangent line at (0,1)

Classifications of DEs

We will use the following two classifications of DEs:

- By order: As we discussed in the previous section, the order of a DE is the order of the highest derivative present in the equation. So, we can talk about DEs of order one, two, three and so on.

- By linearity: A DE of the form

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \ldots + a_1(t)\,y'(t) + a_0(t)\,y(t) = f(t),$$

where the functions $a_n(t)$, ..., $a_0(t)$ are given and act as coefficients of the derivatives of the unknown function and f (t) is the function on the right hand side, is called a linear DEof order n.
DEs in any other form are called non-linear.

Examples.

(1) The DE

$$(t^3 + 1)y''(t) + \sin(t) \cdot y'(t) - 5y(t) = e^t$$

is a linear DE of order 2.

(2) The DE

$$(t^3 + 1)y''(t) + \sin(y'(t)) - 5y(t) = e^t$$

is a non-linear DE of order 2.

(3) The DE

$$y'(t) + y^2(t) = t + 1$$

is non-linear and of first order.

(4) The DE

$$y''' + 3y''(t) \cdot y'(t) - ty(t) = 1$$

is non-linear and of third order.

Examples of DEs modelling real-life phenomena

(1) Radioactive decay

It is known that a radioactive material decomposes at a rate proportional to the amount present at the current time. This can be expressed as a DE

$$M'(t) = kM(t), \quad 0 \le t,$$

where M (t) is the mass of the radioactive material present after time t. As we will see later, the solutions of this first order, linear DE are of the form

$$M(t) = ce^{kt}.$$

The constant k is determined experimentally by the half-life of the radioactive material, while the parameter c is determined by the initial condition

$$M(0) = M_0,$$

which describes the amount of the material present at time t = 0.

(2) Population dynamics.

In 1798 the English economist Thomas Malthus proposed that a population grows at a rate proportional to its size. This leads to the same DE as in the case of radioactive decay:

$$N'(t) = kN(t), \quad t \ge 0.$$

Notice that the radioactive decay has the same DE as this model of population dynamics. However, in the case of the radioactive decay the solution is accurate on long time periods, while in the case of the population dynamics only on a short term, except an idealistic situ- ation of an isolated population with unlimited resources.

For a demonstration of this model see:

http://demonstrations.wolfram.com/ContinuousExponentialGrowth/

In a more realistic scenario, the growth rate depends on the size of the populations as well as on external environmental factors, like limited resources. One possible scenario leads to the logistic DE

$$N'(t) = \alpha N(t)\left(\beta - N(t)\right),$$

where $\beta > 0$ is the carrying capacity of the environment.

For a demonstration of this model see:
http://demonstrations.wolfram.com/LogisticEquation/

If more than one species interact within the same environment, then we need systems to describe their behavior. In case of two animal species, where the first species eats only vegetation and the second species eats the first species, we are lead to the Lotka-Volterra prey-predator model:

$$x'(t) = -ax(t) + bx(t)y(t)$$
$$y'(t) = dy(t) - cx(t)y(t),$$

where a, b, c, d are positive constants and the functions x(t), y(t) describe the number of the population of the two species.

For a demonstration of the two species model check:
http://demonstrations.wolfram.com/PredatorPreyModel/

For a more realistic model see:
http://demonstrations.wolfram.com/PredatorPreyEcosystemARealTimeAgentBasedSimulation/

(3) Series RLC electric circuits.
The DE describing the state of an electric circuit comes from Kirchhoff's second law of electricity, which says that the sum of the voltage drops around the circuit must add up to the electromotive force. In case of a circuit containing an inductor, a capacitor and a resistor, we denote by L, R, C the inductance, resistance and capacitance. The DE describing this circuit is

$$L\,q''(t) + R\,q'(t) + \frac{1}{C}q(t) = E(t),$$

where q(t) is the charge on the capacitor and E(t) is the impressed voltage at time t.

For a demonstration of a series RLC circuit check:
http://demonstrations.wolfram.com/SeriesRLCCircuits/

(4) Mass-Spring systems.
The DE describing a vertical, free mass-spring system follows from Hooke's law and has the form

$$my''(t) + ky(t) = 0, \quad t \geq 0,$$

where y(t) is the the vertical displacement measured from the natural length of the spring,
m is the mass attached to the spring and k is the proportionality constant of the spring. However, if we assume that damping forces proportional to the velocity act on the mass- spring system, then we have the DE

$$my''(t) + \delta y'(t) + ky(t) = 0,$$

where $\delta > 0$ is the damping constant.
To have unique solutions, we have to give, as initial conditions, the initial height and the initial velocity at which the spring is released.

For a demonstration on this problem check:
http://demonstrations.wolfram.com/FreeVibrationsOfASpringMassDamperSystem/

First order differential equations solvable by analytical methods

In this chapter we present several types of first order DEs, which can be solved by algebraic manipulations and integrations.

3.1. Differential equations with separable variables

DEs with separable variables have the form

$$y'(t) = f(t) \cdot g(y(t)).$$

We simplify the way we write these equations in order to separate the variables:

$$y' = f(t) \cdot g(y).$$

Then replace y' by $\frac{dy}{dt}$

$$\frac{dy}{dt} = f(t) \cdot g(y),$$

and get

$$\frac{dy}{g(y)} = f(t)\,dt .$$

Integrate the left side with respect to y and the right side with respect to t to obtain an equation of the form

$$G(y) = F(t) + c .$$

This is the implicit form of the solution. Solving this equation in y gives the solution in explicit form.

Examples.

(1) Solve the DE

$$y' = \frac{t}{5 \cdot y}, \quad -5 < t <$$

Solution:

$$\frac{dy}{dt} = \frac{t}{y}$$

$$y\,dy = t\,dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + c$$

$y^2 = t^2 + c$, solution in implicit form

$y(t) = \pm\sqrt{t^2 + c}$, two families of solutions.

(2) Solve the IVP

$$y' = \frac{t}{y}, \quad y(0) = 2$$

First we solve the DE as in Example 1 and get

$$y(t) = \pm\sqrt{t^2 + c}.$$

The initial condition shows that we have to use the family of solutions with negative sign and get

$$y(0) = -\sqrt{c} = -2,$$

which gives c = 4. Therefore, the solution is

$$y(t) = -\sqrt{t^2 + 4}.$$

(3) Solve the DE

$$y' = t\sqrt{y}, \quad t \in R.$$

For separating the variables we need to divide the DE by $\sqrt{y}$, which possibly excludes the constant function $y(t) \equiv 0$ from the family of solutions we get. However, if we substitute the constant 0 function into the DE, we get the identity $0 \equiv 0$, which shows that $y(t) \equiv 0$ is a solution. Later we will see that it is a singular solution.

$$\frac{dy}{\sqrt{y}} = t\,dt$$

$$2\sqrt{y} = \frac{t^2}{2} + c$$

$$y(t) = \left(\frac{t^2}{4} + \frac{c}{2}\right)^2$$

Observing that $\frac{c}{2}$ is just playing the role of an arbitrary constant, to simplify the form of the solutions, we can replace it by c. In conclusion, we have the one-parameter family of solutions

$$y(t) = \left(\frac{t^2}{4} + c\right)^2.$$

In this family no particular value of c gives the constant 0 function, hence $y(t) \equiv 0$ is not member of this family, and therefore it is a singular solution.

Solving DEs and IVPs with "Mathematica".

In this section we solve the DE $y'(t) = 2ty(t)$ analytically. The solutions of DEs by numerical methods will be shown in Section 4.4.

Start with the Mathematica input line:

which means that the family of solutions is

$$y(t) = ce^{t^2}.$$

If we want to solve the IVP

$$y'(t) = 2ty(t), \quad y(1) = 2,$$

then we use the input line

```
DSolve[{y'[t] == 2*t*y[t],y[1]==2}, y[t], t].
```

The answer is

```
y[t] -> 2e^(-1+t2)
```

which means that the solution is

$$y(t) = 2e^{-1+t^2} = \frac{2}{e}e^{t^2},$$

and hence $c = \frac{2}{e}$.

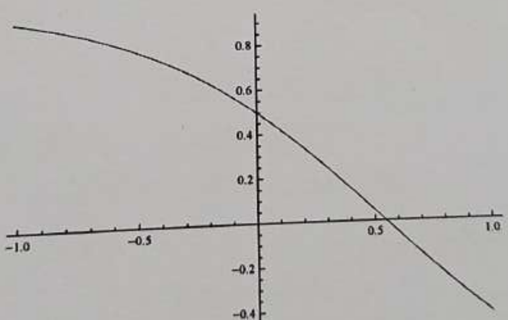
If we want to solve and graph the solution of the IVP

$$y'(t) = y^2(t) - 1, \quad y(2) = 1,$$

then we use the lines:

```
sol = DSolve[{y'[t] == (y[t])^2 - 1, y[0] == 0.5}, y[t], t]
Plot[Evaluate[y[t] /.   sol], {t, -1, 1}]
```

The graph is:



. Graph the solution curve. What is $\lim_{t\to\infty} y(t)$?

First order linear differential equations

The first order linear differential equations have the general form of

$$a(t)y'(t) + b(t)y(t) = f(t). \qquad (3.2.1)$$

If the function f on the right hand side is constantly 0, then we say that the equation is homogeneous. Otherwise, it is non-homogeneous. The following steps are required to solve a first order linear DE:

Step 1.
Given a non-homogeneous linear DE (3.2.1), first we solve the corresponding homogeneous DE

$$a(t)y'(t) + b(t)y(t) = 0. \qquad (3.2.2)$$

We solve it as a separable DE.

$$a(t)y' = -b(t)y$$

$$\frac{dy}{y} = -\frac{b(t)}{a(t)}. \qquad (3.2.3)$$

Let's stop for a moment. The division by y, shows that, as in the previous section, we have to check, by substitution into (3.2.2), that the constant function y(t) 0 is a solution. Indeed, it is, but as we will see later that it is not a singular solution, because it is a member of the family of solutions we get.

Also, the division by a(t), shows that the domain of the solutions has to exclude the numbers t for which a(t) becomes 0.

Using the notation

$$u(t) + c = \int -\frac{b(t)}{a(t)}\,dt$$

the integration of (3.2.3) leads to

$$\ln|y| = u(t) + c.$$

By exponentiating both sides we get that

$$e^{\ln|y|} = e^{u(t)+c} = e^{u(t)} \cdot e^{c},$$

and by replacing the positive constant e^c to a general constant c, we get that

$$y(t) = c\, e^{u(t)}.$$

In conclusion, the family of solutions of the homogeneous linear DE (3.2.2) always has the general form

$$y_h(t) = c\, z(t).$$

Note that, for c = 0, the constant 0 function is a member of this family of solutions.

Step 2.
We need a so-called particular solution of the non-homogeneous linear DE, which will be found by the variation of parameters method. We search for the particular solution as

$$y_p(t) = c(t)z(t),$$

where c(t) is an unknown function and z(t) is taken from Step 1. Substitute $y_p(t)$ into the non-homogeneous equation (3.2.1):

$$a(t)\left[c'(t)z(t) + c(t)z'(t)\right] + b(t)c(t)z(t) = f(t).$$

Rearrange this equation as

$$a(t)c'(t)z(t) + c(t)\left[a(t)z'(t) + b(t)z(t)\right] = f(t),$$

and use the fact that z(t) is a solution of the homogeneous equation, which makes the expression inside the square brackets be 0. Hence,

$$c'(t) = \frac{f(t)}{a(t)z(t)},$$

and therefore c(t) is an antiderivative of $\frac{f(t)}{a(t)z(t)}$. Once c(t) is determined, we get $y_p(t)$.

Step 3.
Finally, the solution of the non-homogeneous linear DE (3.2.1) looks like

$$y(t) = y_h(t) + y_p(t).$$

Note. This method is not valid for non-linear differential equations. In particular, it cannot be used to solve the DE $y' + ty^2 = t$.

Example. Solve the DE

$$y' - 2ty = t.$$

Step 1.

$$y' - 2ty = 0$$

$$\frac{dy}{dt} = 2ty$$

$$\frac{dy}{y} = 2t\,dt$$

$$\ln|y| = t^2 + c$$

$$|y| = e^{t^2+c}$$

$$y(t) = ce^{t^2}$$

Step 2.

$$y_p(t) = c(t)e^{t^2}$$
$$y_p'(t) = c'(t)e^{t^2} + c(t)2te^{t^2}$$
$$c'(t)e^{t^2} + c(t)2te^{t^2} - 2tc(t)e^{t^2} =$$
$$t\ c'(t)e^{t^2} = t$$
$$c'(t) = te^{-t^2}$$
$$c(t) = \int te^{-t^2}dt = -\frac{1}{2}e^{-t^2}$$
$$y_p(t) = -\frac{1}{2}e^{-t^2}e^{t^2} = -\frac{1}{2}$$

Step 3.

$$y(t) = ce^{t^2} - \frac{1}{2}.$$

Bernoulli's differential equations

Bernoulli's differential equations have the form

$$y' + a(t)y = b(t)y^k,$$

where $k \neq 0$ and $k \neq 1$. This is a non-linear equation, which will be changed to a linear one.

Changing the non-linear DE into a linear DE.
Divide the equation by y^k and get

$$y^{-k} y' + a(t)y^{1-k} = b(t).$$

Introduce a new function

$$z(t) = y^{1-k}(t),$$

for which

$$z'(t) = (1-k) \cdot y^{-k}(t) \cdot y'(t).$$

Therefore, the non-linear Bernoulli's DE is changed to

$$\frac{1}{1-k} z' + a(t)z = b(t),$$

which is a first order linear DE in the unknown function z(t).

Solve the first order linear DE in z(t).
This is done according to the Steps 1, 2 and 3 from the previous section.

Return to y(t). Write

$$y(t) = z(t)^{\frac{1}{1-k}},$$

which is the solution of the Bernoulli's DE.

Example. Solve the DE

$$y' + \frac{1}{t}y = t^2y^2, \quad t > 0.$$

Solution:
Changing the non-linear DE into a linear DE.
Divide the DE by y^2:

$$y^{-2}y' + \frac{1}{t}y^{-1} = t^2.$$

Introduce

$$z(t) = (y(t))^{-1} = \frac{1}{y(t)}.$$

Then, $z' = (-1)y^{-2}y'$ and the linear DE in z looks like

$$z' + \frac{1}{t}z = t^2.$$

Solve the first order linear DE in z(t). Step 1.

$$z' + \frac{1}{t}z = 0$$

$$\frac{dz}{z} = \frac{dt}{t}$$

$$\ln |z| = \ln |t| + c$$

$$z_h(t) = c\,t\,.$$

Step 2. Search the particular solution in the form $z_p(t) = c(t) \cdot t$. By substituting $z_p(t)$ into the DE of $z(t)$ gives $c'(t) = -t$, which gives $c(t) = -\frac{t^2}{2}$ and hence

$$z_p(t) = -\frac{t^3}{2}\,.$$

Step 3.

$$z(t) = ct - \frac{t^3}{2}\,.$$

Return to $y(t)$.

$$y(t) = \frac{1}{ct - \frac{t^3}{2}}\,.$$

Non-linear homogeneous differential equations

The non-linear part of the title has the meaning to distinguish between the earlier studied linear homogeneous DEs and the ones in this section. Note, that, while most of the DEs in this section are non-linear, there are linear DEs which are homogeneous in this non-linear sense.

The non-linear homogeneous differential equations have the form

$$y' = f\left(\frac{y}{t}\right)$$

We can solve them by introducing a new function

$$z(t) = \frac{y(t)}{t}$$

Hence,

$$y(t) = tz(t)$$

and

$$y' = z + tz'$$

The new DE in z is

$$z + tz' = f(z),$$

which is always a DE with separable variable. After solving this DE in z, we can get y(t) from the equation y(t) = t z(t).

Example. Solve the DE

$$t^2y' - y^2 - yt = 0, \quad t > 0.$$

Solution:
Dividing the equation by t^2 gives:

$$y' = \left(\frac{y}{t}\right)^2 + \frac{y}{t}.$$

Then,

$$z = \frac{y}{t}$$

$$y = tz$$

$$y' = z + tz'$$

$$z + tz' = z^2 + z$$

$$t\frac{dz}{dt} = z^2$$

$$\frac{dz}{z^2} = \frac{dt}{t}, \quad z \neq 0$$

Note: z(t)= 0 is excluded from the solutions, so we have to check, by substitution, whether it is a solution or not. It turns out that it is a solution.

$$\frac{-1}{z} = \ln t$$

Not that z(t) = 0 is not part of this family, so it is a singular solution. Therefore, the solutions of this problem can be organized in a one-parameter family of solutions

$$y = \frac{-t}{\ln t + c}$$

and a singular solution

$$y(t) \equiv 0 .$$

Second order differential equations reducible to first order differential equations

We will solve second order differential equations which contain just y'' and y', and no y. These equations have the general form form $f(t, y', y'') = 0$.
If we introduce the function $z = y'$, then we get a first order DE in z: $f(t, z, z') = 0$. Once we get z, the solution y is found by integration.

Example.
Solve the IVP:

$$y'' + 3y' = e^{2t}, \quad y(0) = 1, \; y'(0) = 0.$$

Solution:
Introducing the function $z = y'$ we get the linear DE in z:

$$z' + 3z = e^{2t}.$$

Solving this equation in z gives:

$$z(t) = ce^{-3t} + \frac{1}{5}e^{2t}.$$

Integrating z leads to

$$y(t) = \frac{-c}{3}e^{-3t} + \frac{1}{10}e^{2t} + d.$$

The initial conditions give the system

$$-\frac{c}{3} + \frac{1}{10} + d = 1$$
$$c + \frac{1}{5} = 0.$$

Solving this system in c and d gives $c = -\frac{1}{5}$ and $d = \frac{5}{6}$.
Therefore, the solution of the IVP is

$$y(t) = \frac{1}{15}e^{-3t} + \frac{1}{10}e^{2t} + \frac{5}{6}.$$

General theory of differential equations of first order

Slope fields (or direction fields)

Consider a first order DE in normal form

$$y^{J}(t) = \mathbf{f}(t, y(t)),\ t \in I .$$

If $y : I R$ is a solution to this DE, then at any point t_0 I, the value of $\mathbf{f}$ $(t_0, y(t_0))$ is the slope to the graph of the function y, which is a solution curve to the DE.

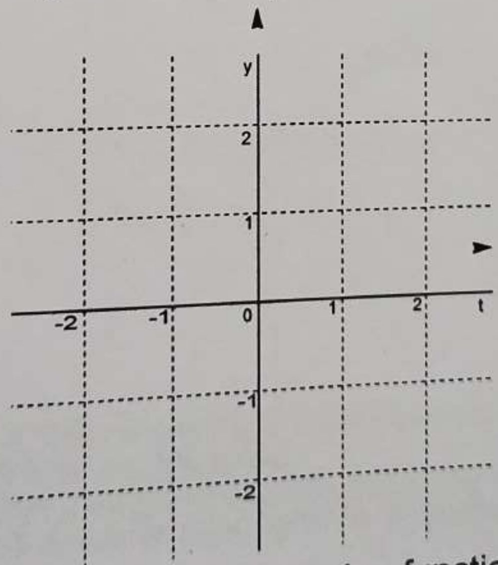
Therefore, if we show a rectangular grid in the ty-coordinate system and evaluate $\mathbf{f}$ (t, y) at the points in the grid, then we have graphical information about where solution curves are heading, without actually solving the DE.

DEFINITION. A slope field of a DE is a rectangular grid with slopes, as arrows pointing left, drawn at each point of the grid.

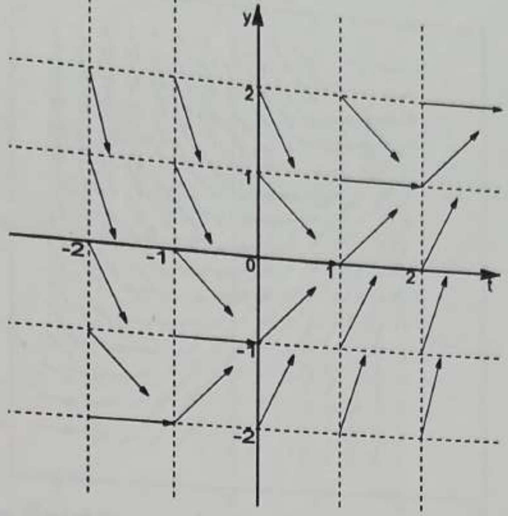
Example. This example shows how to draw a slope field manually. Consider the DE

$$y^{J} = t - y .$$

Draw first a grid in the ty-coordinate system for t = −2, −1, 0, 1, 2 and y = −2, −1, 0, 1, 2

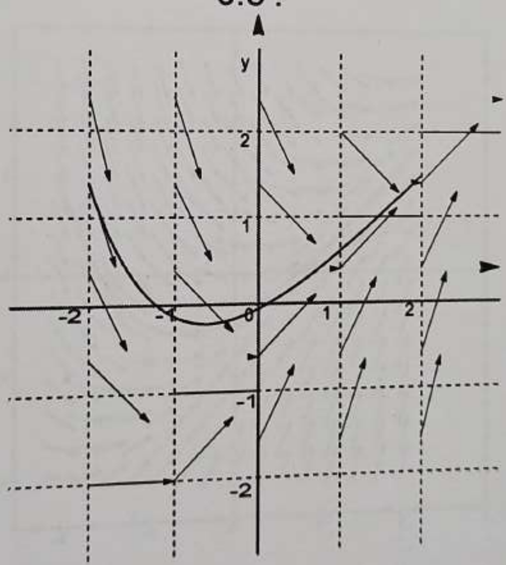


The right hand side to the DE gives the function $\mathbf{f}$ (t, y) = t y. Evaluate this function at each point of the grid and show the results as slopes at the corresponding points. For example, $\mathbf{f}$ (2, 1) = 1 gives a slope 1 at the point (2, 1). Continuing in this way we get the following slope field.



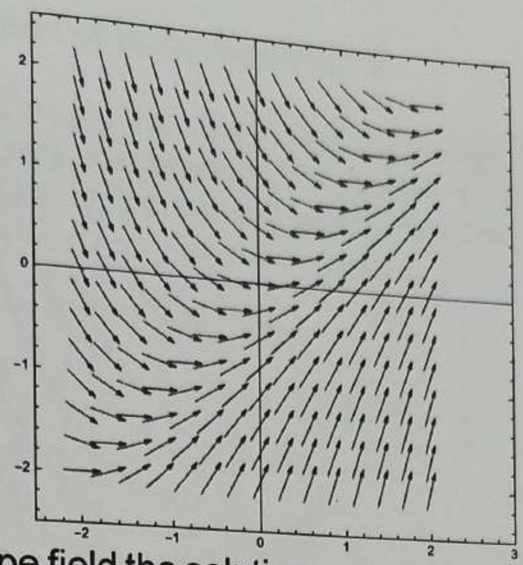
Based on the slope field we can get graphical information about solution curves. If we choose an initial point, then we can draw an approximative solution curve on the graph by following the slopes in the slope field. The following graph shows the slope field and solution curve for the IVP

- y' = t -
y y(−1)
= 0.5 .



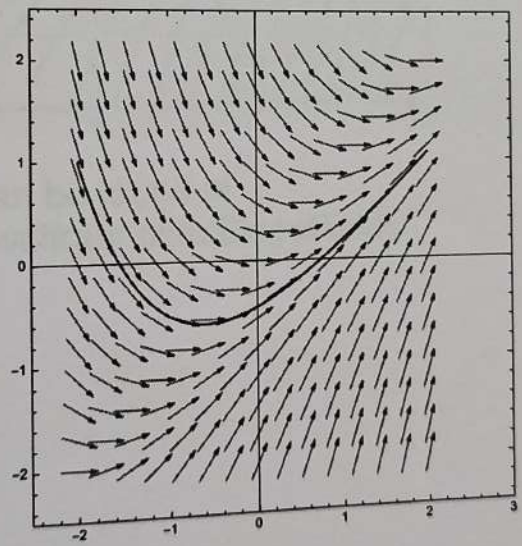
Of course, if the slope field is filled with more slopes, our information about solution curves is more complete.
Mathematica can graph a slope field in the following way. The role of the cosine arctangent and the sine arctangent is to restrict the length of each vector to one.

```
VectorPlot[{Cos[ArcTan[t - y]], Sin[ArcTan[t - y]] , }  {t, -2, 2 ,}
{y, -2, 2 ,}PlotRange-> {{ -2.5, 3 }  {-2.5, 2.5 }} , Axes ->
True, VectorStyle -> Arrowheads[0.02]]
```



We can add to the slope field the solution curve starting at (2, 1), which shows how solution curves follow the slopes.

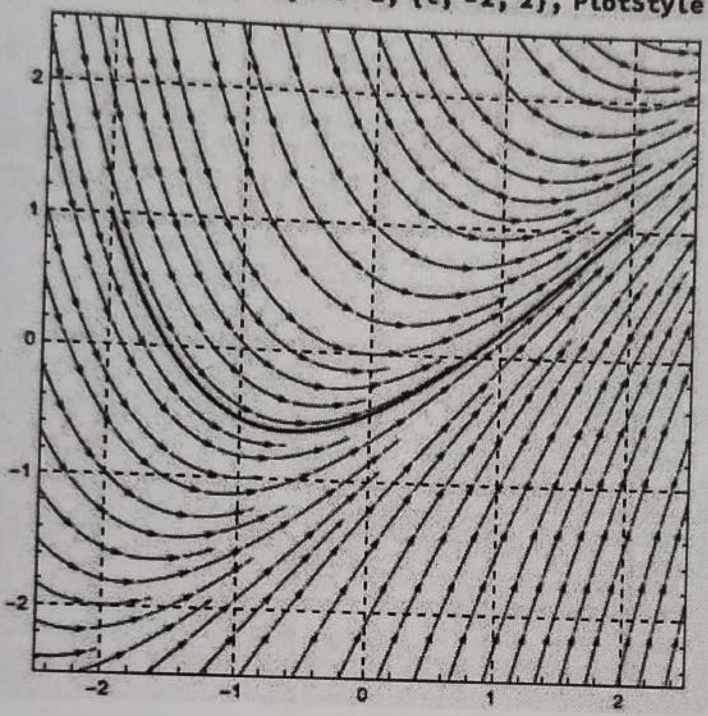
```
Show[VectorPlot[{ Cos[ArcTan[t - y]], Sin[ArcTan[t - y]] }, {t, -2, 2 },
{y, -2, 2 }, PlotRange -> {{ -2.5, 3 }, {-2.5, 2.5 }} , Axes->True,
VectorStyle -> Arrowheads[0.015]], Plot[4*Exp[-t - 2] + t - 1, {t, -2, 2 }
, PlotStyle -> Red]]
```



Also, there is the option of using StreamPlot.

```
In[50]:= Show[StreamPlot[{1, t - y}, {t, -2.5, 2.5}, {y, -2.5, 2.5}, GridLines → Automatic,
    GridLinesStyle → Directive[Dashed], PlotRange → {{-2.5, 2.5}, {-2.5, 2.5}}],
  Plot[4 * Exp[-t - 2] + t - 1, {t, -2, 2}, PlotStyle → Red]]
```

Out[50]=



More slope fields can be found at http://demonstrations.wolfram.com/SlopeFields/.

Autonomous first order differential equations.

First order DEs in the form

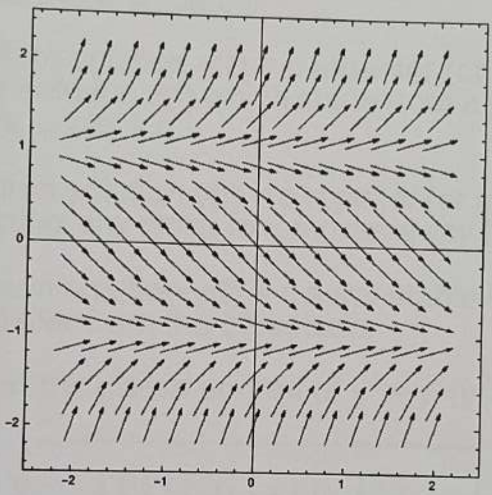
$$y'(t) = f(y(t)),$$

or shortly

$$y' = f(y),$$

are called autonomous first order DEs. Their slope fields show equal slopes along horizontal grid lines. For example, lets have a look at the slope field of

$$y' = y^2 - 1.$$



DEFINITION. A phase portrait for a first order DE is a slope field with several solution curves, showing the most important qualitative properties of solutions.

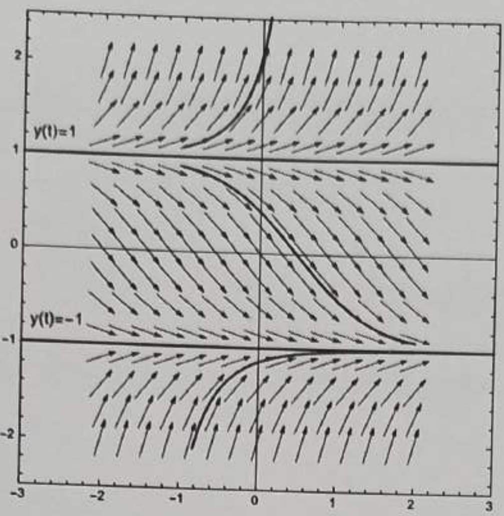
DEFINITION Critical numbers (or points) for an autonomous first order DE are numbers c such that $f(c) = 0$.

DEFINITION Equilibrium solutions are the constant functions $y(t) = c$, corresponding to the critical numbers c.

Example. Consider the DE

$$y' = y^2 - 1.$$

In this case $f(y) = y^2 - 1$ and the critical numbers correspond to the solutions of $y^2 - 1 = 0$, which are ±1. Hence the critical numbers are $c = -1$ and $c = 1$, while the equilibrium solutions are $y(t) = -1$ and $y(t) = 1$. The phase portrait in this case looks like:



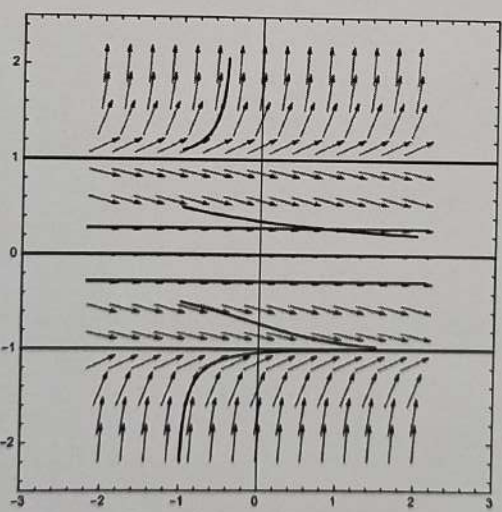
Classifications of equilibrium solutions:

(a) We call an equilibrium solution y(t) = c attractor (or asymptotically stable) if for any other solution z(t) which starts from a position sufficiently close to c, we have $\lim_{t\to\infty} z(t) = c$.

(b) We call an equilibrium solution y(t) = c repeller (or unstable) if any other solution z(t) starting any close to c moves away from it as $t \to \infty$.

(c) We call an equilibrium solution y(t) = c semi-stable if it is an attractor from one side and repeller from the other side.

Example. Let us look at the phase portrait of $y' = y^2(y^2 - 1)$.



The y(t) = 1 is a repeller, y(t) = 0 is semi-stable and y(t) = −1 is anattractor

Higher order linear differential equations

General theory

A n^{th}-order linear DE has the form

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t),\quad t\in I, \qquad (5.1.1)$$

where the unknown function is $y(t)$ and the coefficients are the functions $a_k(t)$, $0\le k\le n$.

Example. In the case of

$$(t^3-1)y^{(4)}(t)+\sqrt{t^2+4}\,y'''(t)-\sin t\, y'(t)+y(t)=e^t,\quad 1<t<\infty,$$

$a_4(t)=t^3-1$, $a_3(t)=\sqrt{t^2+4}$, $a_2(t)=0$, $a_1(t)=-\sin t$, $a_0(t)=1$ and $g(t)=e^t$.

The general solution of a n^{th}-order linear DE has the form

$$y(t)=y_h(t)+y_p(t),$$

where $y_h(t)$ is a n-parameter family of solutions of the linear and homogeneous DE

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=0,\quad t\in I, \qquad (5.1.2)$$

and $y_p(t)$ is a particular solution of the non-homogeneous DE (5.1.1). As a n-parameter family of solutions, $y_h(t)$ has to be determined as

$$y_h(t)=c_1y_1(t)+\cdots+c_ny_n(t),$$

where $y_1(t),\cdots y_n(t)$ are solutions of the linear and homogeneous DE (5.1.2). However, not every choice of n solutions is suitable. We must choose linearly independent solutions, which means that if

$$c_1y_1(t)+\cdots+c_ny_n(t)=0,\quad \text{for every } t\in I,$$

then each parameter must be 0:

$$c_1=\cdots=c_n=0.$$

To analytically check the linear independence of solutions, we must check the Wronskian determinant is not identically zero:

$$W(y_1(t),y_2(t),\cdots y_n(t))=\begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t)\\ y_1'(t) & y_2'(t) & \cdots & y_n'(t)\\ \vdots & \vdots & \cdots & \vdots\\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t)\end{vmatrix}\neq 0,$$

for at least one $t \in I$.

Note: Determinants are calculated in the following way:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Higher order determinants are calculated in a similar way by expanding them using the first row, and thus reducing the calculations to determinants of one size less.

Definition. The functions $y_1(t), \cdots, y_n(t)$ form a Fundamental Set of Solutions (shortly FSS) of the linear and homogeneous DE (5.1.2) if:

1. Each function is a solution.
2. They are linearly independent.

The following theorem gives us a method to check whether n functions form a FSS or not.

THEOREM 5.1.1. If the functions $y_1(t), \cdots, y_n(t)$ are solutions of the linear and homogeneous DE (5.1.2) and $W(y_1(t), \cdots, y_n(t)) \neq 0$ for at least one $t \in I$, then they are linearly independent and form a FSS.

Examples:

(1) Let us show that the functions $y_1(t) = t$ and $y_2(t) = t^3$ form a FSS for the DE

$$t^2 y'' - 3t\, y' + 3y = 0, \quad t \in (0, +\infty).$$

First, let us check that the two functions are solutions. By substituting $y_1(t) = t$ into the DE we get

$$t^2 \cdot 0 - 3t \cdot 1 + 3t = 0,$$

which leads to $0 = 0$. Repeat the process for $y_2(t) = t^3$, too. Then

$$W(t, t^3) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 3t^3 - t^3 = 2t^3,$$

which is not zero for any (would be enough to check just for one) $t > 0$. Therefore, $y_1(t) = t$ and $y_2(t) = t^3$ form a FSS.

However, if we want to see whether $z_1(t) = t$ and $z_2 = 5t$ form a FSS, then we can check that they are solutions, but

$$W(t, 5t) = \begin{vmatrix} t & 5t \\ 1 & 5 \end{vmatrix} = 5t - 5t = 0,$$

which shows that they are not linearly independent. Therefore, they do not form a FSS.

Regarding the existence and uniqueness of solutions for IVPs

THEOREM . Consider the IVP

$$\begin{cases} a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = g(t)\ ,\ t \in [\alpha, \beta] \\ y(t_0) = y_0,\ y'(t_0) = y_1, \cdots , y^{(n-1)}(t_0) = y_{n-1}, \end{cases}$$

where $t_0 \in [\alpha, \beta]$ is a fixed point.
If the functions $a_n(t), \cdots, a_0(t), g(t)$ are continuous on the interval $[\alpha, \beta]$ and $a_n(t)$
any $\alpha \leq t \leq \beta$, then the IVP has a unique solution on the entire interval $[\alpha, \beta]$.

Step 1. Using the method from Section 5.2 we obtain $y_h(t) = c_1 + c_2e^t$.

Step 2. In this exercise $g(t) = (2t + 3)e^{0\,t}$ and $\alpha = 0$, which is a simple ($k = 1$) solution of the characteristic equation $r^2 - r = 0$. So, we search for $y_p(t)$ in the form

$$y_p(t) = t(b_1t + b_0)e^{0t} = b_1t^2 + b_0t\,.$$

Substituting $y_p(t)$ into the DE leads to

$$2b_1 - 2b_1t - b_0 = 2t + 3\,,$$

which can be rearranged as

$$-2b_1t + 2b_1 - b_0 = 2t + 3\,.$$

The two sides must be identically the same, so we have $-2b_1 = 2$ and $2b_1 - b_0 = 3$, which gives $b_1 = -1$ and $b_0 = 5$ and hence $y_p(t) = -t^2 - 5t$.

Step 3. The final form of the solution is

$$y(t) = c_1 + c_2e^t - t^2 - 5t\,.$$

5.1. The Cauchy-Euler DE

The Cauchy-Euler DE has the form

$$a_n \cdot t^n \cdot y^{(n)}(t) + a_{n-1} \cdot t^{n-1} \cdot y^{(n-1)}(t) + \cdots + a_1 \cdot t \cdot y'(t) + a_0 \cdot y(t) = g(t)\,, \qquad (5.4.1)$$

which has to be solved for $t < 0$ or $t > 0$. This is a linear DE with non-constant coefficients and we will reduce it to a linear DE with constant coefficients. In order to achieve this, we use the substitutions

$$t = e^x \text{ or } x = \ln t\,, \text{ if } t > 0\,,$$

and

$$t = -e^x \text{ or } x = \ln(-t)\,, \text{ if } t < 0.$$

Let us consider the $t > 0$ case.

We have to substitute the derivatives in t with derivatives in x. Using the chain rule we get that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t} = \frac{dy}{dx}e^{-x}.$$

Furthermore,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \left(\frac{d}{dx}\left(\frac{dy}{dx}e^{-x}\right)\right)e^{-x} = \left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)e^{-2x}\,,$$

and

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = \left(\frac{d}{dx}\left(\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)e^{-2x}\right)\right)e^{-x} = \left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)e^{-3x}.$$

If we continue in this way, we can express any derivative in t in terms of derivatives in x and by substituting them into the equation (5.4.1), we obtain a linear differential equation with constant coefficients.

Example Solve the following DE:

$$t^3y''' + 5t^2y'' + 7ty' + 8y = 2\ln t\,, \quad t > 0.$$

Let us use the substitution $t = e^x$ and get

$$e^{3x}\left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)e^{-3x} + 5e^{2x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)e^{-2x} + 7e^{x}\frac{dy}{dx}e^{-x} + 8y = 2x$$

Analytical solutions of Differential Equations

We will use "DSolve" to get an analytical solution to the DE y'(t) = 2ty(t).

In[1]:= DSolve[y' [t]==2*t*y[t],y [t], [t]

Out[1]= { { y [t] -> e^{t2} C [1] } }

The answer corresponds to the one patrameter family of solutions y(t) = c e^{t2}. Let's solve noe the IVP y'(t)=2ty(t), y(1)=2.

In[2]:= DSolve[{ y ' [t] ❖ 2* t* y [t], y [1] == 2}, y [t] ,[t]

Out[2]= { { y [t] → $2e^{-1+t2}$ } }

The answer corresponds to the solution y(t) = 2 $e^{-1}e^{t2} = \frac{2}{e} e^{t2}$.

If we want to plot the solution, first we have to define the solution as a function:

In[3]:= sol = DSolve [{ y' [t] == 2* t* y [t] , y [1]==2}, y [t] , t]

Out[3]= { { y [t] → $2e^{-1+t2}$ } }

In[4]:= z [t_] := Evaluate [y [t] /. sol]

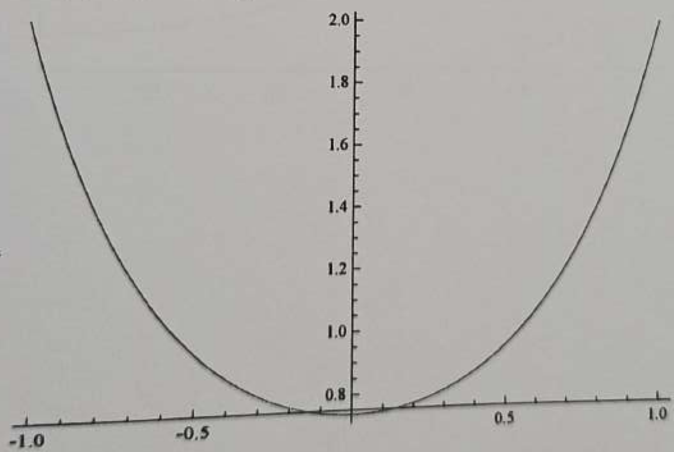
Now, z(t) is the solution function and we can use it for evaluation and graphing:

In[6]:= z [0.1]

Out[6]= { 0.743153

In[7]:= Plot [z[t], { t, -1, 1 }]

Out[7]=



DUDHNOI COLLEGE

PROJECT TITLE

LIMIT & CONTINUITY

SUBMITTED TO -

Department of Mathematics

Dudhnoi College, Dudhnoi.

Examined by
Mridul Dutta

SUBMITTED BY -

Name : Parves Musharof

Roll No.:- US-191-097-0048

Reg. No.- 19023512

DECLARATION

I, PARVES MUSHAROF hereby declare that the project entitled "LIMIT & CONTINUITY" submitted to Department of Mathematics, Dudhnoi College, Dudhnoi for the degree of Bachelor of Science in Mathematics in faculty of science is a record of original work done by me under the supervision of Ripunjoy Choudhury, Assistant professor, Department of Mathematics, Dudhnoi College. I would like to declare that neither the project nor any part there of has been submitted to this college/institute or elsewhere for the award of any other degree or diploma.

Parves Musharof

(Parves Musharof)

Date: 18-07-2022

Place: Dudhnoi

Roll No:US-191-097-0048

Department of Mathematics,

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I sincerely thanks to my classmate and my family for their help, advice and encouragement in carrying out the project.

Place: Dudhnoi

Date:18-07-2022

Parwes Musharob

Signature

CERTIFICATE

This is to certify that the project entitled "Limit & Continuity" is the out come of the study and investigations carried out by Parves Musharof. It has been done under my supervision and guidance and neither the project nor any part there of has been submitted to this or any other college for a bachelor of science degree.

Mridul Dutta 18/7/22

Dr.Mridul dutta

Date:18-07-2022

Assistant professor

Place: Dudhnoi

Department of Mathematics,Dudhnoi college

Limits and continuity

Assumed knowledge

The content of the modules:

- *Algebra review*
- *Functions I*
- *Functions II*
- *Coordinate geometry*.

Motivation

Functions are the heart of modelling real-world phenomena. They show explicitly the relationship between two (or more) quantities. Once we have such a relationship, various questions naturally arise.

For example, if we consider the function

$$f(x) = \frac{\sin x}{x},$$

we know that the value $x = 0$ is not part of the function's domain. However, it is natural to ask: What happens *near* the value $x = 0$? If we substitute small values for x (in radians), then we find that the value of $f(x)$ is approximately 1. In the module *The calculus of trigonometric functions*, this is examined in some detail. The closer that x gets to 0, the closer the value of the function $f(x) = \frac{\sin x}{x}$ gets to 1.

Another important question to ask when looking at functions is: What happens when the independent variable becomes very large? For example, the function $f(t) = e^{-t} \sin t$ is used to model damped simple harmonic motion. As t becomes very large, $f(t)$ becomes very small. We say that $f(t)$ approaches zero as t goes to infinity.

Both of these examples involve the concept of **limits**, which we will investigate in this module. The formal definition of a limit is generally not covered in secondary school mathematics. This definition is given in the *Links forward* section. At school level, the notion of limit is best dealt with informally, with a strong reliance on graphical as well as algebraic arguments.

When we first begin to teach students how to sketch the graph of a function, we usually begin by plotting points in the plane. We generally just take a small number of (generally integer) values to substitute and plot the resulting points. We then 'join the dots'. That we can 'join the dots' relies on a subtle yet crucial property possessed by many, but not all, functions; this property is called **continuity**. In this module, we briefly examine the idea of continuity.

Content

Limit of a sequence

Consider the sequence whose terms begin

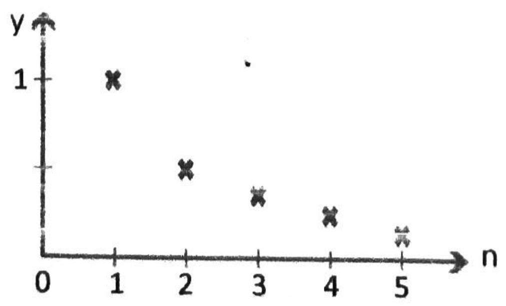
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

and whose general term, is $\frac{1}{n}$. As we take more and more terms, each term is getting smaller in size. Indeed, we can make the terms as small as we like, provided we go far enough along the sequence. Thus, although no term in the sequence is 0, the terms can be made as close as we like to 0 by going far enough.

We say that the limit of the sequence $\frac{1}{n}$; $n = 1, 2, 3, \ldots$ is 0 and we write

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

It is important to emphasise that we are not putting n equal to ∞ in the sequence, since infinity is not a number — it should be thought of as a convenient idea. The statement above says that the terms in the sequence $\frac{1}{n}$ get as close to 0 as we please (and continue to be close to 0), by allowing n to be large enough.



Graph of the sequence $\frac{1}{n}$.

In a similar spirit, it is true that we can write

$$\lim_{n\to\infty} \frac{1}{n^a} = 0,$$

for any positive real number a. We can use this, and some algebra, to find more complicated limits.

Example

Find

$$\lim_{n\to\infty} \frac{3n^2 + 2n + 1}{n^2 - 2}.$$

Solution

Intuitively, we can argue that, if n is very large, then the largest term (sometimes called the **dominant** term) in the numerator is $3n^2$, while the dominant term in the denominator is n^2. Thus, ignoring the other terms for the moment, for very large n the expression $\frac{3n^2+2n+1}{n^2-2}$ is close to 3.

The best method of writing this algebraically is to divide by the highest power of n in the denominator:

$$\lim_{n\to\infty} \frac{3n^2 + 2n + 1}{n^2 - 2} = \lim_{n\to\infty} \frac{3 + \frac{2}{n} \cdot \frac{1}{n^2}}{1 - \frac{2}{n^2}}.$$

Now, as n becomes as large as we like, the terms $\frac{2}{n}$, $\frac{1}{n^2}$ and $\frac{2}{n^2}$ approach 0, so we can complete the calculation and write

$$\lim_{n\to\infty} \frac{3n^2 + 2n + 1}{n^2 - 2} = \lim_{n\to\infty} \frac{3 + \frac{2}{n} \cdot \frac{1}{n^2}}{1 - \frac{2}{n^2}}$$

$$= \frac{\lim_{n\to\infty} 3 + \frac{2}{n} + \frac{1}{n^2}}{\lim_{n\to\infty} 1 - \frac{2}{n^2}}$$

$$= \frac{3}{1} = 3.$$

Exercise 1

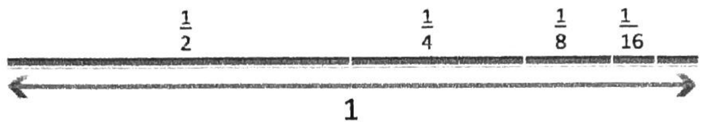
Find

$$\lim_{n\to\infty} \frac{5n^3 + (-1)^n}{4n^3 + 2}.$$

Limiting sums

A full study of infinite series is beyond the scope of the secondary school curriculum. But one infinite series, which was studied in antiquity, is of particular importance here.

Suppose we take a unit length and divide it into two equal pieces. Now repeat the process on the second of the two pieces, and continue in this way as long as you like.



Dividing a unit length into smaller and smaller pieces.

This generates the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Intuitively, the sum of all these pieces should be 1.

After n steps, the distance from 1 is $\frac{1}{2^n}$. This can be written as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}.$$

The value of the sum approaches 1 as n becomes larger and larger. We can write this as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} \to 1 \quad \text{as} \quad n \to \infty.$$

We also write this as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1.$$

This is an example of an infinite geometric series.

A series is simply the sum of the terms in a sequence. A **geometric sequence** is one in which each term is a constant multiple of the previous one, and the sum of such a sequence is called a **geometric series**. In the example considered above, each term is $\frac{1}{2}$ times the previous term.

A typical geometric sequence has the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

where $r \neq 0$. Here a is the **first term**, r is the constant multiplier (often called the **common ratio**) and n is the number of terms.

The terms in a geometric sequence can be added to produce a geometric series:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \qquad (1)$$

We can easily find a simple formula for S_n. First multiply equation (1) by r to obtain

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n. \qquad (2)$$

Subtracting equation (2) from equation (1) gives

$$S_n - rS_n = a - ar^n$$

from which we have

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad \text{for } r \neq 1.$$

Now, if the common ratio r is less than 1 in magnitude, the term r^n will become very small as n becomes very large. This produces a **limiting sum**, sometimes written as S_∞. Thus, if $|r| < 1$,

$$S_\infty = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

In the example considered at the start of this section, we have $a = \frac{1}{2}$ and $r = \frac{1}{2}$, hence the value of the limiting sum is $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$, as expected.

Exercise 2

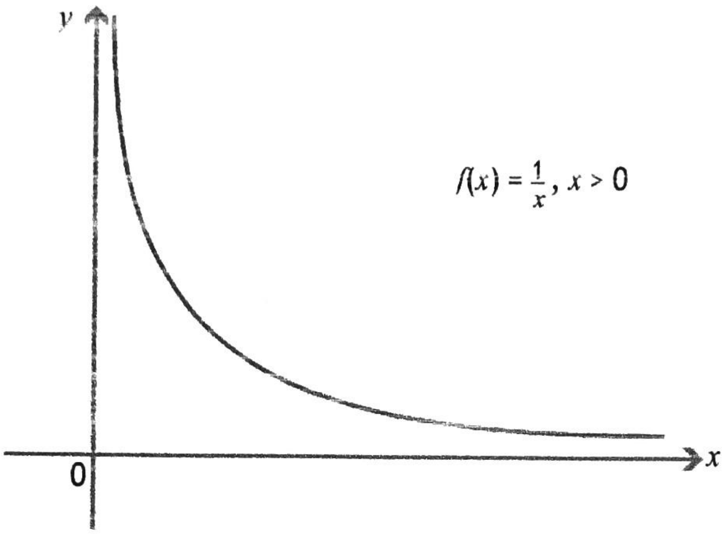
Find the limiting sum for the geometric series

$$\frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \cdots$$

Limit of a function at infinity

Just as we examined the limit of a sequence, we can apply the same idea to examine the behaviour of a function $f(x)$ as x becomes very large.

For example, the following diagram shows the graph of $f(x) = \frac{1}{x}$, for $x > 0$. The value of the function $f(x)$ becomes very small as x becomes very large.

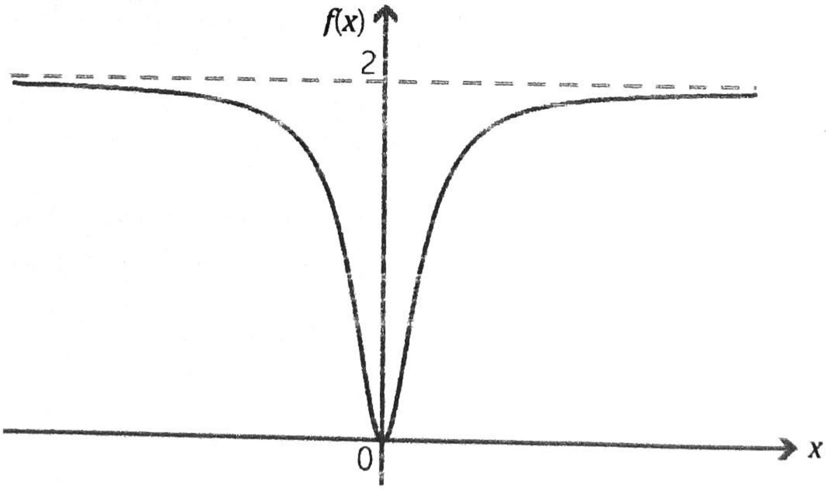


We write

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

One of the steps involved in sketching the graph of a function is to consider the behaviour of the function for large values of x. This will be covered in the module *Applications of differentiation*.

The following graph is of the function $f(x) = \frac{2x^2}{1 + x^2}$. We can see that, as x becomes very large, the graph levels out and approaches, but does not reach, a height of 2.



We can analyse this behaviour in terms of limits. Using the idea we saw in the section *Limit of a sequence*, we divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2}{1+x^2} = \lim_{x\to\infty} \frac{2}{\frac{1}{x^2}+1} = 2.$$

Note that as x goes to negative infinity we obtain the same limit. That is,

$$\lim_{x\to-\infty} \frac{2x^2}{1+x2} = 2.$$

This means that the function approaches, but does not reach, the value 2 as x becomes very large. The line $y = 2$ is called a **horizontal asymptote** for the function.

Exercise 3

Find the horizontal asymptote for the function $f(x) = \dfrac{x^2-1}{3x^2+1}$.

Examining the *long-term* behaviour of a function is a very important idea. For example, an object moving up and down under gravity on a spring, taking account of the inelasticity of the spring, is sometimes referred to as **damped simple harmonic motion**. The displacement, $x(t)$, of the object from the centre of motion at time t can be shown to have the form

$$x(t) = Ae^{-\alpha t} \sin \beta t,$$

where A, α and β are positive constants. The factor $Ae^{-\alpha t}$ gives the amplitude of the motion. As t increases, this factor $Ae^{-\alpha t}$ diminishes, as we would expect. Since the factor $\sin \beta t$ remains bounded, we can write

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} Ae^{-\alpha t} \sin \beta t = 0.$$

In the long term, the object returns to its original position.

Limit at a point

As well as looking at the values of a function for large values of x, we can also look at what is happening to a function near a particular point.

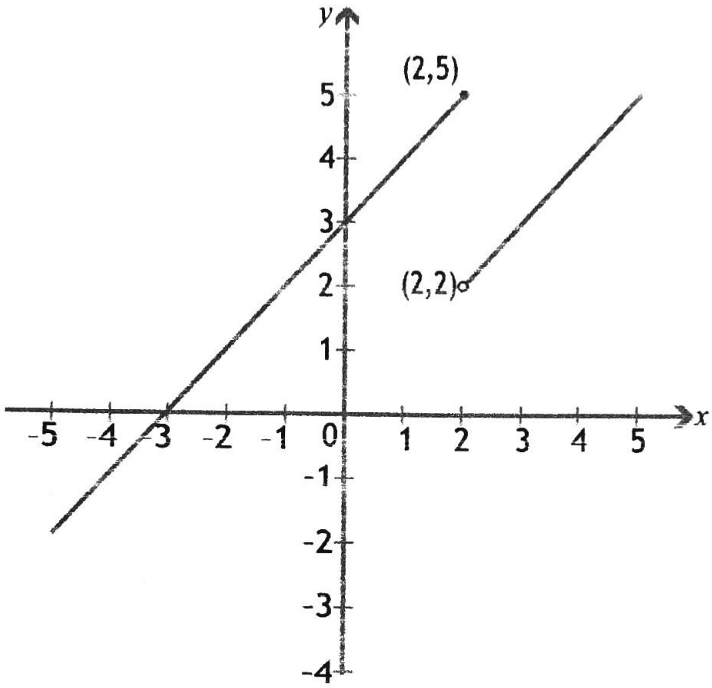
For example, as x gets close to the real number 2, the value of the function $f(x) = x^2$ gets close to 4. Hence we write

$$\lim_{x\to 2} x^2 = 4.$$

Sometimes we are given a function which is defined piecewise, such as

$$f(x) = \begin{cases} x + 3 & \text{if } x \le 2 \\ x & \text{if } x > 2. \end{cases}$$

The graph of this function is as follows.



We can see from the 'jump' in the graph that the function does not have a limit at 2:

- as the x-values get closer to 2 from the left, the y-values approach 5
- but as the x-values get closer to 2 from the right, the y-values do not approach the same number 5 (instead they approach 2).

In this case, we say that

$$\lim_{x \to 2} f(x) \text{ does not exist.}$$

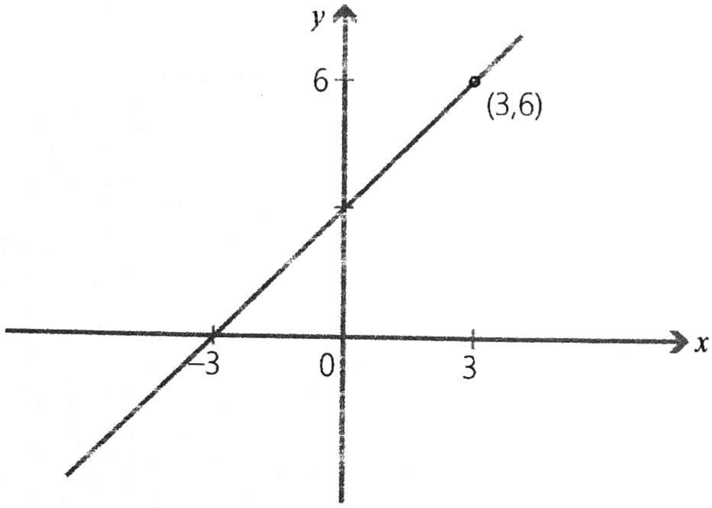
Sometimes we are asked to analyse the limit of a function at a point which is not in the domain of the function. For example, the value $x = 3$ is not part of the domain of the function $f(x) = \frac{x^2 - 9}{x - 3}$. However, if $x \neq 3$, we can simplify the function by using the difference of two squares and cancelling the (non-zero) factor $x - 3$:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3, \qquad \text{for } x \neq 3.$$

Now, when x is near the value 3, the value of $f(x)$ is near $3 + 3 = 6$. Hence, near the x-value 3, the function takes values near 6. We can write this as

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

The graph of the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is a straight line with a hole at the point $(3,6)$.



Exampie

Find

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 4}.$$

Solution

We cannot substitute $x = 2$, as this produces 0 in the denominator. We therefore factorise and cancel the factor $x - 2$:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)(x - 1)}{(x - 2)(x + 2)}$$

$$= \lim_{x \to 2} \frac{x - 1}{x + 2} = \frac{1}{4}.$$

Even where the limit of a function at a point does not exist, we may be able to obtain useful information regarding the behaviour of the function near that point, which can assist us in drawing its graph.

For example, the function

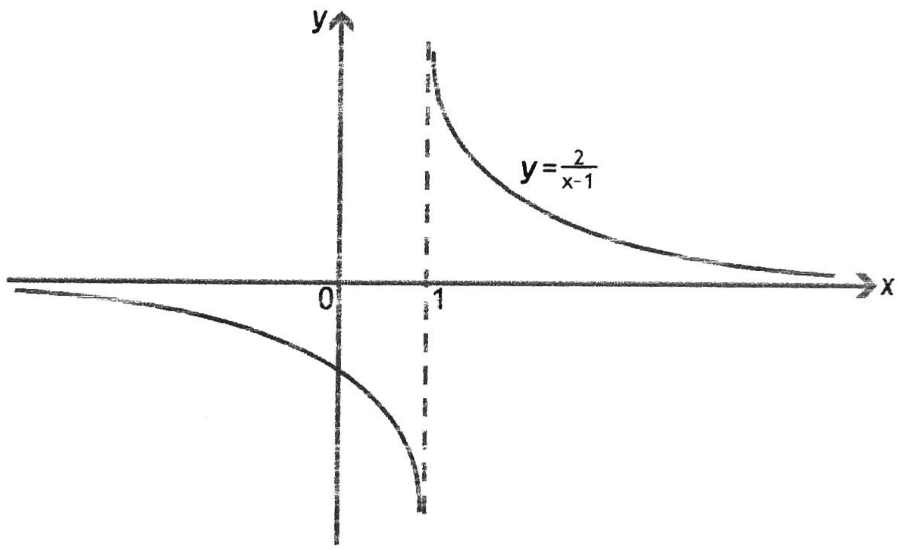
$$f(x) = \frac{2}{x-1}$$

is not defined at the point $x = 1$. As x takes values close to, but *greater than* 1, the values of $f(x)$ are very large and positive, while if x takes values close to, but *less than* 1, the values of $f(x)$ are very large and negative. We can write this as

$$\frac{2}{x-1} \to \infty \text{ as } x \to 1^+ \quad \text{and} \quad \frac{2}{x-1} \to -\infty \text{ as } x \to 1^-.$$

The notation $x \to 1^+$ means that 'x approaches 1 from above' and $x \to 1^-$ means 'x approaches 1 from below'.

Thus, the function $f(x) = \frac{2}{x-1}$ has a vertical asymptote at $x = 1$, and the limit as $x \to 1$ does not exist. The following diagram shows the graph of the function $f(x)$. The line $y = 0$ is a horizontal asymptote.



Exercise 4

Discuss the limit of $f(x) = \frac{x^2}{x^2 - 1}$ at the points $x = 1$, $x = -1$ and as $x \to \pm\infty$.

Exercise 5

Discuss the limit of the function

$$f(x) = \frac{x^2 - 2x - 15}{2x^2 + 5x - 3} = \frac{(x-5)(x+3)}{(2x-1)(x+3)}$$

as

a $x \to \infty$ **b** $x \to 5$ **c** $x \to -3$ **d** $x \to \frac{1}{2}$ **e** $x \to 0$

Further examples

There are some examples of limits that require some 'tricks'.

For example, consider the limit

$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}.$$

We cannot substitute $x=0$, since then the denominator will be 0. To find this limit, we need to rationalise the numerator:

$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}=\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\times\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}$$

$$=\lim_{x\to 0}\frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}=\frac{1}{4}.$$

Exercise 6

Find

a $\displaystyle\lim_{x\to 1}\frac{\sqrt{x^2+15}-4}{x-1}$ **b** $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}$.

So far in this module, we have implicitly assumed the following facts — none of which we can prove without a more formal definition of limit.

Algebra of limits

Suppose that $f(x)$ and $g(x)$ are functions and that a and k are real numbers. If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

a $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

b $\lim_{x\to a} kf(x)=k\lim_{x\to a} f(x)$

c $\lim_{x\to a} f(x)g(x)=\lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$

d $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $\lim_{x\to a} g(x)$ *is not equal to* 0.

Continuity

When first showing students the graph of $y = x^2$, we generally calculate the squares of a number of x-values and plot the ordered pairs (x, y) to get the basic shape of the curve. We then 'join the dots' to produce a *connected* curve.

We can justify this either by plotting intermediary points to show that our plot is reasonable or by using technology to plot the graph. That we can 'join the dots' is really the consequence of the mathematical notion of **continuity**.

A formal definition of continuity is not usually covered in secondary school mathematics. For most students, a sufficient understanding of *continuity* will simply be that they can draw the graph of a continuous function without taking their pen off the page. So, in particular, for a function to be continuous at a point a, it must be defined at that point.

Almost all of the functions encountered in secondary school are continuous everywhere, unless they have a good reason not to be. For example, the function $f(x) = \frac{1}{x}$ is continuous everywhere, except at the point $x = 0$, where the function is not defined.

A point at which a given function is not continuous is called a **discontinuity** of that function.

Here are more examples of functions that are continuous everywhere they are defined:

- polynomials (for instance, $3x^2 + 2x - 1$)
- the trigonometric functions $\sin x$, $\cos x$ and $\tan x$
- the exponential function a^x and logarithmic function $\log_b x$ (for any bases $a > 0$ and $b > 1$).

Starting from two such functions, we can build a more complicated function by either adding, subtracting, multiplying, dividing or composing them: the new function will also be continuous everywhere it is defined.

Example

Where is the function $f(x) = \frac{1}{x^2 - 16}$ continuous?

Solution

The function $f(x) = \frac{1}{x^2 - 16}$ is a quotient of two polynomials. So this function is continuous everywhere, except at the points $x = 4$ and $x = -4$, where it is not defined.

Continuity of piecewise-defined functions

Since functions are often used to model real-world phenomena, sometimes a function may arise which consists of two separate pieces joined together. Questions of continuity can arise in these case at the point where the two functions are joined. For example, consider the function

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3. \end{cases}$$

This function is continuous everywhere, except possibly at $x = 3$. We can see whether or not this function is continuous at $x = 3$ by looking at the limit as x approaches 3. Using the ideas from the section *Limit at a point*, we can write

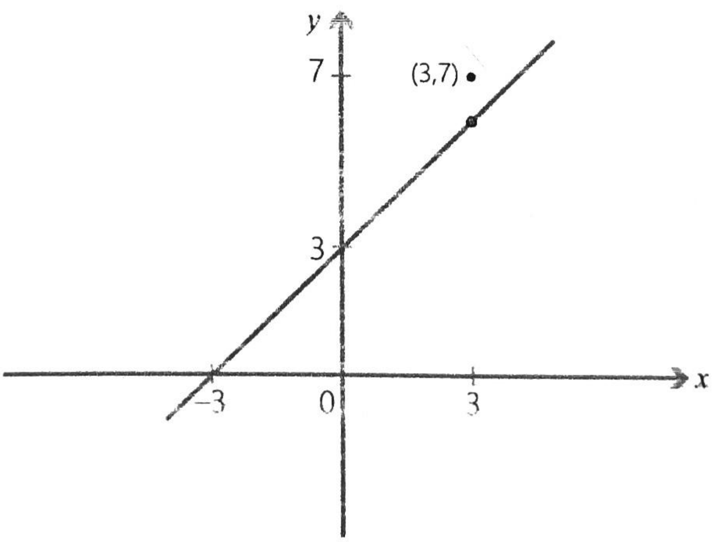
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = 6.$$

Since 6 is also the value of the function at $x = 3$, we see that this function is continuous. Indeed, this function is identical with the function $f(x) = x + 3$, for all x.

Now consider the function

$$g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3. \end{cases}$$

The value of the function at $x = 3$ is different from the limit of the function as we approach 3, and hence this function is not continuous at $x = 3$. We can see the discontinuity at $x = 3$ in the following graph of $g(x)$.



We can thus give a slightly more precise definition of a function $f(x)$ being continuous at a point a. We can say that $f(x)$ is **continuous** at $x = a$ if

- $f(a)$ is defined, and
- $\lim_{x \to a} f(x) = f(a)$.

Example

Examine whether or not the function

$$f(x) = \begin{cases} x^3 - 2x + 1 & \text{if } x \le 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

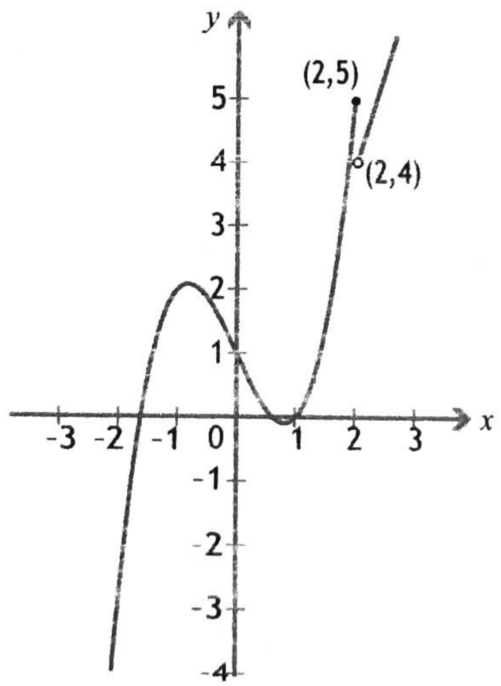
is continuous at $x = 2$.

Solution

Notice that $f(2) = 2^3 - 2 \times 2 + 1 = 5$. We need to look at the limit from the right-hand side at $x = 2$. For $x > 2$, the function is given by $3x - 2$ and so

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2) = 4.$$

In this case, the limit from the right at $x = 2$ does not equal the function value, and so the function is not continuous at $x = 2$ (although it is continuous everywhere else).



Exercise 7

Examine whether or not the function

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \le 0 \\ 4 + x & \text{if } x > 0 \end{cases}$$

is continuous

Links forward

Formal definition of a limit

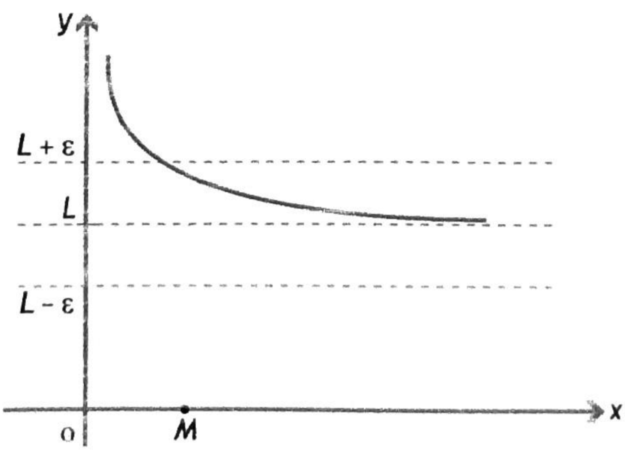
In this module, the notion of limit has been discussed in a fairly informal manner. To be able to prove results about limits and capture the concept logically, we need a formal definition of what we mean by a limit. We will only look here at the precise meaning of $\lim_{x\to\infty} f(x)$ L, but there is a similar definition for the limit at a point.

In words, the statement $\lim_{x\to\infty} f(x) = L$ says that $f(x)$ gets (and stays) as close as we please to L, provided we take sufficiently large x. We now try to pin down this notion of *closeness*.

Another way of expressing the statement above is that, if we are given any small positive number ε (the Greek letter *epsilon*), then the distance between $f(x)$ and L is less than ε provided we make x large enough. We can use absolute value to measure the distance between $f(x)$ and L as $|f(x)-L|$.

How large does x have to be? Well, that depends on how small ε is.

The formal definition of $\lim_{x\to\infty} f(x) = L$ is that, given any $\varepsilon > 0$, there is a number M such that, if we take x to be larger than M, then the distance $|f(x)-L|$ is less than

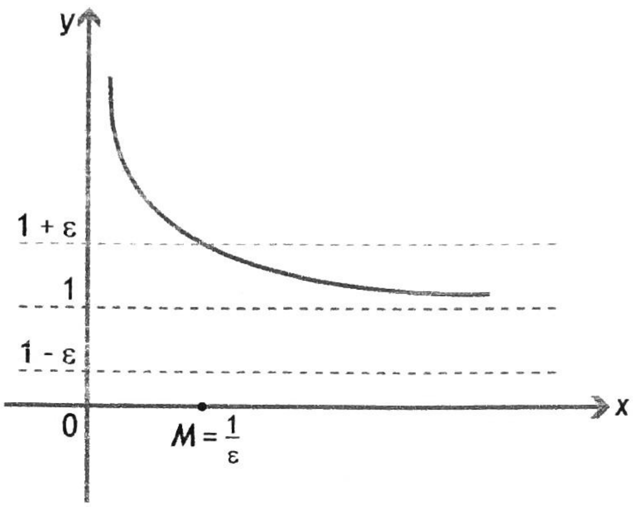


The value of $f(x)$ stays within ε of L from the point $x = M$ onwards.

For example, consider the function $f(x) = \frac{x+1}{x}$. We know from our basic work on limits that $\lim_{x\to\infty} f(x) = 1$. For $x > 0$, the distance is

$$|f(x) - 1| = \left|\frac{x+1}{x} - 1\right| = \frac{1}{x}.$$

So, given any positive real number ε, we need to find a real number M such that, if $x > M$, then $\frac{1}{x} < \varepsilon$. For $x > 0$, this inequality can be rearranged to give $x > \frac{1}{\varepsilon}$. Hence we can choose M to be $\frac{1}{\varepsilon}$.



Exercise 8

Let $f(x) = \frac{2x^2+3}{x^2}$. Given $\varepsilon > 0$, find M such that if $x > M$ we have $|f(x)-2| < \varepsilon$. Conclude that $f(x)$ has a limit of 2 as $x \to \infty$.

While the formal definition can be difficult to apply in some instances, it does give a very precise framework in which mathematicians can properly analyse limits and be certain about what they are doing.

The pinching theorem

One very useful argument used to find limits is called the **pinching theorem**. It essentially says that if we can 'pinch' our limit between two other limits which have a common value, then this common value is the value of our limit.

Thus, if we have

$$g(x) \le f(x) \le h(x), \qquad \text{for all } x,$$

and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Here is a simple example of this.

To find $\lim_{n\to\infty} \frac{n!}{n^n}$, we can write

$$\frac{n!}{n^n} = \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{3}{n} \times \frac{2}{n} \times \frac{1}{n}$$

$$\leq 1 \times 1 \times 1 \times \cdots \times 1 \times 1 \times \frac{1}{n} = \frac{1}{n},$$

where we replaced every fraction by 1 except the last. Thus we have $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we can conclude using the pinching theorem that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Other examples will be found in later modules. In particular, the very important limit

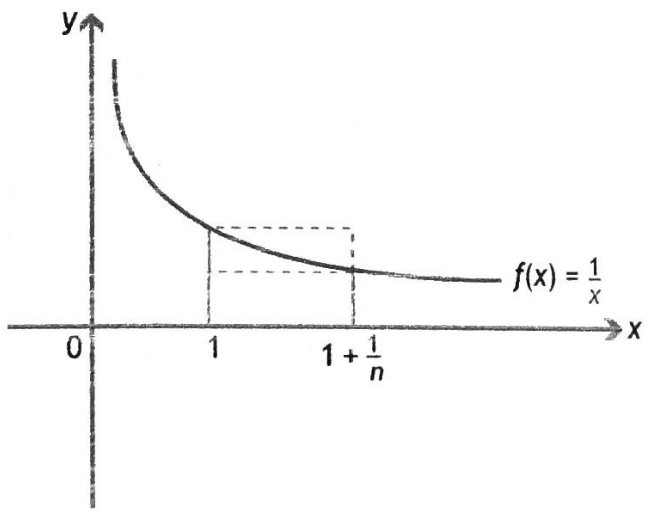
$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0$$

(where x is expressed in radians) will be proven using the pinching theorem in the module *The calculus of trigonometric functions*.

Finding limits using areas

One beautiful extension of the pinching theorem is to bound a limit using areas.

We begin by looking at the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 1 + \frac{1}{n}$.



The area under the curve is bounded above and below by areas of rectangles, so we have

$$\frac{1}{n} \times \frac{1}{1+\frac{1}{n}} \leq \int_1^{1+\frac{1}{n}} \frac{1}{x}\, dx \leq \frac{1}{n} \times 1.$$

Hence

$$\frac{1}{1+n} \le \log_e\left(1+\frac{1}{n}\right) \le \frac{1}{n}.$$

Now multiplying by n, we have

$$\frac{n}{1+n} \le n\log_e\left(1+\frac{1}{n}\right) \le 1.$$

Hence, if we take limits as $n \to \infty$, we conclude by the pinching theorem that

$$n\log_e\left(1+\frac{1}{n}\right) \to 1 \implies \log_e\left(1+\frac{1}{n}\right)^n \to 1$$

$$\implies \left(1+\frac{1}{n}\right)^n \to e \qquad (!!)$$

That is,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

History and applications

Paradoxes of the infinite

The ancient Greek philosophers appear to have been the first to contemplate the infinite in a formal way. The concept worried them somewhat, and Zeno came up with a number of paradoxes which they were not really able to explain properly. Here are two of them:

The dichotomy paradox

Suppose I travel from A to B along a straight line. In order to reach B, I must first travel half the distance AB_1 of AB. But to reach B_1 I must first travel half the distance AB_2 of AB_1, and so on ad infinitum. They then concluded that motion is impossible since, presumably, it is not possible to complete an infinite number of tasks.

The paradox of Achilles and the tortoise

A tortoise is racing against Achilles and is given a head start. Achilles is much faster than the tortoise, but in order to catch the tortoise he must reach the point P_1 where the tortoise started, but in the meantime the tortoise has moved to a point P_2 ahead of P_1. Then when Achilles has reached P_2 the tortoise has again moved ahead to P_3. So on ad infinitum, and so even though Achilles is faster, he cannot catch the tortoise.

In both of these supposed paradoxes, the problem lies in the idea of adding up infinitely many quantities whose size becomes infinitely small.

Pi as a limit

The mathematician François Vieta (1540–1603) gave the first theoretically precise expression for π, known as Vieta's formula:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \cdots.$$

This expresses π as the limit of an infinite product.

John Wallis (1616–1703), who was one of the most influential mathematicians in England in the time just prior to Newton, is also known for the following very beautiful infinite product formula for π:

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times \cdots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times \cdots}.$$

He also introduced the symbol ∞ into mathematics.

Infinitesimals

The notion of an infinitesimal essentially goes back to Archimedes, but became popular as a means to explain calculus. An infinitesimal was thought of as an infinitely small but non-zero quantity. Bishop Berkeley (1685–1753) described them as the 'ghosts of vanished quantities' and was opposed to their use.

Unfortunately, such quantities do not exist in the real number system, although the concept may be useful for discovering facts that can then be made precise using limits.

The real number system is an example of an **Archimedean system**: given any real number α, there is an integer n such that $n\alpha > 1$. This precludes the existence of infinitesimals in the real number system.

Non-archimedean systems can be defined, which contain elements which do not have this property. Indeed, all of calculus can be done using a system of mathematics known as **non-standard analysis**, which contains both infinitesimals and infinite numbers.

Cauchy and Weierstrass

Prior to the careful analysis of limits and their precise definition, mathematicians such as Euler were experimenting with more and more complicated limiting processes; sometimes finding correct answers — often for wrong reasons — and sometimes finding incorrect ones. A lack of rigour often led to paradoxes of the type we looked in the section *Paradoxes of the infinite*.

In the early 19th century, the need for a more formal and logical approach was beginning to dawn on mathematicians such as Cauchy and later Weierstrass.

The French mathematician Augustine-Louie Cauchy (pronounced Koshi, with a long o) (1789–1857) was one of the early pioneers in a more rigorous approach to limits and calculus. He was also responsible for the development of **complex analysis**, which applies the notions of limits and calculus to functions of a complex variable. Many theorems and equations in that subject bear his name. Cauchy is regarded by many as a pioneer of the branch of mathematics known as analysis, although he also made use of infinitesimals.

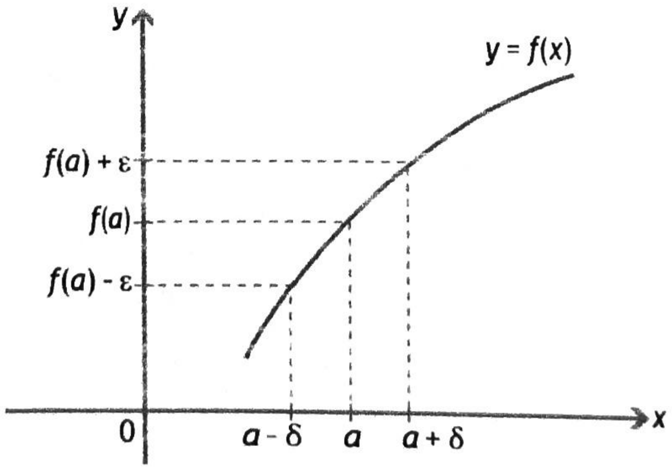
Analysis may be thought of as the theoretical side of limits and calculus. It is a very important branch of modern mathematics, and teaches us how to deal with calculus in ways that are rigorous and logically valid.

In 1821, Cauchy wrote *Cours d'Analyse*, which had a great impact on continental mathematics. In it he introduced proofs using the ε notation we saw in the section *Links forward* (*Formal definition of a limit*).

At roughly the same time, Bernard Bolzano (1781–1848) was attempting to deal with some of the classical paradoxes in his book *The paradoxes of the infinite*. He was the first to give a rigorous ε-δ definition of a limit, although much of his work was not widely disseminated at the time.

While Cauchy made mathematicians think more deeply about what they were doing, it was Karl Weierstrass (1815–1897) who is generally regarded as the father of modern analysis. He gave the first rigorous definition of continuity of a function $f(x)$ at a point a.

The definition states: Given $\varepsilon > 0$, there is a positive real number δ such that, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.



Continuity of $f(x)$ at $x = a$.

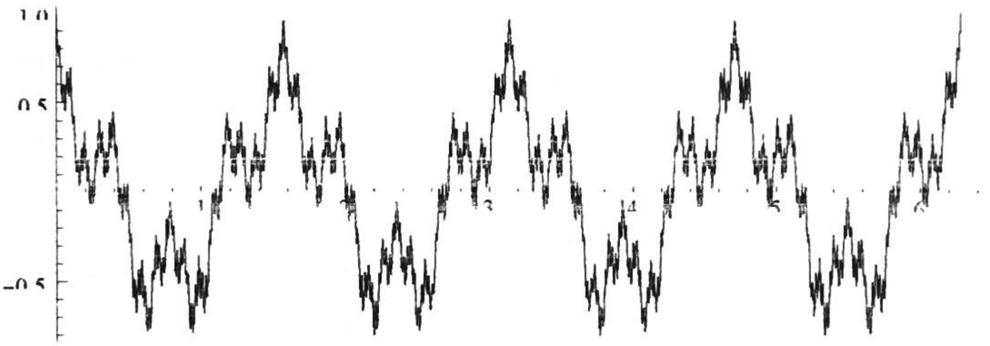
This basically says that a function $f(x)$ is continuous at a point a if x-values that are close to a (i.e., within δ of a) get mapped by f to y-values that are close to $f(a)$ (i.e., within ε of $f(a)$).

In fact, this definition of the continuity of $f(x)$ at a says exactly that $\lim_{x \to a} f(x) = f(a)$, using the formal definition of a limit. So it agrees with our definition of continuity in the section *Continuity of piecewise-defined functions*.

Weierstrass's work was very influential and formed a solid foundation for analysis for decades to come. He shocked the mathematical world by coming up with a function which is continuous everywhere but differentiable nowhere! That is, its graph could be drawn without lifting the pen, but it is not possible to draw a tangent to the curve at any point on it! The function he gave is expressed in terms of an infinite series of functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(4^n x).$$

The following diagram, created using Mathematica, gives the graph of $y = \sum_{n=1}^{50} \frac{1}{2^n} \cos(4^n x)$ for $0 \le x \le 2\pi$.



An approximation to Weierstrass's function.

Roughly speaking, this infinite series has trigonometric terms with amplitude $\frac{1}{2^n}$, which quickly approaches 0 as n gets larger. It is this aspect that is used to prove continuity. It can be formally proven that the infinite series converges for every value of x, and that the function so generated is continuous everywhere. If, however, we were to try to differentiate this function term-by-term, then the derivative of the general term is $-2^n \sin(4^n x)$ whose amplitude is 2^n, which becomes very large as n increases. So the series of derivatives does not converge. Hence the function is not differentiable anywhere.

Answers to exercises

Exercise 1

$$\lim_{n\to\infty}\frac{5n^3+(-1)^n}{4n^3+2}=\lim_{n\to\infty}\frac{5+\frac{(-1)^n}{n^3}}{4+\frac{2}{n^3}}=\frac{5}{4}.$$

Exercise 2

In the geometric series $\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\cdots$, the first term a is $\frac{3}{2}$ and the common ratio r is $\frac{3}{4}$ (which is less than 1 in magnitude). So the limiting sum is

$$S_\infty=\frac{a}{1-r}=\frac{\frac{3}{2}}{1-\frac{3}{4}}=6.$$

Exercise 3

$$\lim_{x\to\infty}\frac{x^2-1}{3x^2+1}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{3+\frac{1}{x^2}}=\frac{1}{3}$$

So the function $f(x)=\dfrac{x^2-1}{3x^2+1}$ has horizontal asymptote $y=\frac{1}{3}$.

Exercise 4

Define $f(x)=\dfrac{x^2}{x^2-1}$. Then

$$f(x)\to-\infty \text{ as } x\to1^- \quad\text{and}\quad f(x)\to+\infty \text{ as } x\to1^+,$$

and so $\lim_{x\to1}f(x)$ does *not* exist. Also,

$$f(x)\to+\infty \text{ as } x\to-1^- \quad\text{and}\quad f(x)\to-\infty \text{ as } x\to-1^+,$$

hence $\lim_{x\to-1}f(x)$ does *not* exist. We can calculate

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1}{1-\frac{1}{x^2}}=1 \quad\text{and}\quad \lim_{x\to-\infty}f(x)=1.$$

Exercise 5

Define $f(x)=\dfrac{(x-5)(x+3)}{(2x-1)(x+3)}$. Then:

a $f(x)\to\frac{1}{2}$ as $x\to\infty$

b $f(x)\to0$ as $x\to5$

c $f(x)\to\frac{8}{7}$ as $x\to-3$

d $f(x)\to-\infty$ as $x\to\frac{1}{2}^+$, and $f(x)\to+\infty$ as $x\to\frac{1}{2}^-$, so $f(x)$ has *no* limit as $x\to\frac{1}{2}$

e $f(x)\to5$ as $x\to0$.

Exercise 6

a We rationalise the numerator:

$$\lim_{x\to 1} \frac{\sqrt{x^2+15}-4}{x-1} = \lim_{x\to 1} \frac{\sqrt{x^2+15}-4}{x-1} \times \frac{\sqrt{x^2+15}+4}{\sqrt{x^2+15}+4}$$

$$= \lim_{x\to 1} \frac{x^2-1}{(x-1)\left(\sqrt{x^2+15}+4\right)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)\left(\sqrt{x^2+15}+4\right)}$$

$$= \lim_{x\to 1} \frac{x+1}{\sqrt{x^2+15}+4} = \frac{1}{4}.$$

b We get rid of the fractions in the numerator:

$$\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4} = \lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4} \times \frac{4x}{4x} = \lim_{x\to 4} \frac{4-x}{4x(x-4)}$$

$$= \lim_{x\to 4} \frac{-(x-4)}{4x(x-4)} = \lim_{x\to 4} -\frac{1}{4x} = -\frac{1}{16}.$$

Exercise 7

Clearly $f(0) = 4$. We need to look at the limit at $x = 0$ from above. For $x > 0$, we have $f(x) = 4 + x$. So $f(x) \to 4$ as $x \to 0^+$. Since this limit is equal to $f(0)$, we conclude that f is continuous everywhere.

Exercise 8

Let $\varepsilon > 0$. We want to find M such that, if $x > M$, then $|f(x) - 2| < \varepsilon$. Note that

$$|f(x) - 2| = \left|\frac{2x^2+3}{x^2} - 2\right| = \frac{3}{x^2}.$$

We want $\frac{3}{x^2} < \varepsilon$, which is equivalent to $x^2 > \frac{3}{\varepsilon}$. Hence, we take $M = \sqrt{\frac{3}{\varepsilon}}$. For all $x > M$, we now have $|f(x) - 2| = \frac{3}{x^2} < \varepsilon$. This tells us that $f(x)$ has a limit of 2 as $x \to \infty$.

References

The following introductory calculus textbook begins with a very thorough discussion of functions, limits and continuity.

- Michael Spivak, *Calculus*, 4th edition, Publish or Perish, 2008.

CONCLUSION

From the above discussion it is clear that the major part of the limit and continuity whose roll interior obviously in the last discussion. I have mentioned theorems related to limit and continuity. I have discussed there example and application in various problem. It can be consider as the basic function of limit and continuity.

I am hopeful and believe that the branch of limit of continuity will be developed with great external to coming future. So that we able to solve more together problem.